Exercises for Lecture 6: Optimal mechanisms.

Problem 1: Optimal Procurement Mechanism

Consider an inverted setting to the one discussed in the lecture: a seller now has an item of some privately known quality $\theta \in [0,1]$. Here θ also equals the seller's valuation of the item, and the buyer's valuation is given by $v(\theta) > \theta$. The *buyer* designs a direct revelation mechanism (k,t) to purchase this item, where $k(\theta)$ is the probability of trade and $t(\theta)$ is the payment from the buyer to the seller.[1] Players' expected utilities given the seller's true type θ and the seller's report $\hat{\theta}$ are then given by

$$U_S(\hat{\theta}|\theta) = -\theta k(\hat{\theta}) + t(\hat{\theta}),$$
$$U_B(\hat{\theta}|\theta) = v(\theta) k(\hat{\theta}) - t(\hat{\theta}).$$

1. Show that for an allocation rule $k(\theta)$ to be IC for the seller, it must be weakly *decreasing* in θ.
2. ERP implies that the payoff of the seller of type θ can be written as $U_S(\theta|\theta) = U_S(1|1) + \int_\theta^1 k(s)ds$. Using this expression, show that the buyer's expected utility in any IC DRM (k,t) is given by:
$$\mathbb{E}_\theta[U_B(\theta|\theta)] = \int_0^1 k(\theta) VS(\theta)\phi(\theta)d\theta - U_S(1|1),$$
where $VS(\theta) = v(\theta) - \theta - \frac{\Phi(\theta)}{\phi(\theta)}$.
3. Explain each component of $VS(\theta)$.
4. Suppose from now on that $\theta \sim U[0,1]$. Find the optimal allocation rule when $v(\theta) = \frac{3\theta}{2}$ for $\theta \in [0, \frac{1}{3}]$ and $v(\theta) = \frac{5\theta}{2} - \frac{1}{3}$ for $\theta \in [\frac{1}{3}, 1]$.
5. Find the optimal allocation rule when $v(\theta) = \frac{5\theta}{2}$ for $\theta \in [0, \frac{1}{3}]$ and $v(\theta) = \frac{3\theta}{2} + \frac{1}{3}$ for $\theta \in [\frac{1}{3}, 1]$.

Solution

1. Take two arbitrary seller types θ and $\hat{\theta} > \theta$. The IC condition for θ to not be willing to falsely report $\hat{\theta}$ is

$$\begin{aligned} U_S(\theta|\theta) &\geq U_S(\hat{\theta}|\theta) \\ &\geq -\theta k(\hat{\theta}) + t(\hat{\theta}) \\ &\geq -\hat{\theta} k(\hat{\theta}) + t(\hat{\theta}) + (\hat{\theta} - \theta) k(\hat{\theta}) \\ &\geq U_S(\hat{\theta}|\hat{\theta}) + (\hat{\theta} - \theta) k(\hat{\theta}) \\ \iff k(\hat{\theta}) &\leq -\frac{U_S(\hat{\theta}|\hat{\theta}) - U_S(\theta|\theta)}{\hat{\theta} - \theta}. \end{aligned}$$

Applying a similar chain of reasoning to the IC condition for $\hat{\theta}$ to not be willing to report θ yields

$$k(\theta) \geq -\frac{U_S(\hat{\theta}|\hat{\theta}) - U_S(\theta|\theta)}{\hat{\theta} - \theta},$$

[1] Note that this formulation of the problem explicitly constrains the mechanism to be ex post budget balance requirement.

hence $k(\theta) \geq k(\hat{\theta})$. Types θ and $\hat{\theta} > \theta$ were arbitrary, hence any implementable $k(\theta)$ must be a weakly decreasing function.

2. The expected utility of the buyer can be written as:

$$\mathbb{E}U_B(\theta) \equiv \mathbb{E}U_B(\theta|\theta) = \mathbb{E}_\theta[v(\theta)k(\theta) - t(\theta)]$$

Rearranging U_S to $t(\theta) = U_S(\theta|\theta) + \theta k(\theta)$ and plugging it into the expected utility of the buyer:

$$\mathbb{E}U_B(\theta) = \mathbb{E}_\theta[v(\theta)k(\theta) - U_S(\theta|\theta) - \theta k(\theta)]$$

Now, using the envelope theorem and integration by parts yields:

$$\begin{aligned}
\mathbb{E}U_B(\theta) &= \mathbb{E}_\theta[(v(\theta) - \theta)k(\theta) - U_S(1|1) - \int_\theta^1 k(s)ds] \\
&= \mathbb{E}_\theta\left[(v(\theta) - \theta)k(\theta) - \int_\theta^1 k(s)ds\right] - U_S(1|1) \\
&= \int_0^1 \left[(v(\theta) - \theta)k(\theta) - \int_\theta^1 k(s)ds\right]\phi(\theta)d\theta - U_S(1|1) \\
&= \int_0^1 (v(\theta) - \theta)k(\theta)\phi(\theta)d\theta - \int_0^1 \int_\theta^1 k(s)\phi(\theta)dsd\theta - U_S(1|1) \\
&= \int_0^1 (v(\theta) - \theta)k(\theta)\phi(\theta)d\theta - \int_0^1 k(\theta)\Phi(\theta)d\theta - U_S(1|1) \\
&= \int_0^1 \left(v(\theta) - \theta - \frac{\Phi(\theta)}{\phi(\theta)}\right)k(\theta)\phi(\theta)d\theta - U_S(1|1)
\end{aligned}$$

3. Explain each component of $VS(\theta)$.

$$VS(\theta) = v(\theta) - \theta - \frac{\Phi(\theta)}{\phi(\theta)}$$

$VS(\theta)$ is the virtual surplus gained by the buyer in this model. $v(\theta)$ is the buyer's valuation, θ is the seller's valuation which is equal to the quality of the item which is privately known by the seller. The buyer does not know θ exactly, but he assumes that θ is a random variable with some cumulative distribution function $\Phi(\theta)$ and probability distribution function $\phi(\theta)$.

Term $v(\theta) - \theta$ represents the **surplus** generated by trade, this is what the buyer would maximize if he could extract the information about θ from the seller at no cost. The term $\frac{\Phi(\theta)}{\phi(\theta)}$ is the **information rent** that must be left to the seller in order to incentivize truthtelling. The buyer thus maximizes surplus net of information rents.

4. Calculate virtual surplus $VS(\theta) = v(\theta) - \theta - \frac{\Phi(\theta)}{f(\theta)}$:

$$VS(\theta) = \begin{cases} \frac{3\theta}{2} - \theta - \frac{\theta}{1} & \text{if } \theta \in [0, \frac{1}{3}] \\ \frac{5\theta}{2} - \frac{1}{3} - \theta - \frac{\theta}{1} & \text{if } \theta \in [\frac{1}{3}, 1] \end{cases}$$

$$\Leftrightarrow VS(\theta) = \begin{cases} -\frac{\theta}{2} & \text{if } \theta \in [0, \frac{1}{3}] \\ \frac{\theta}{2} - \frac{1}{3} & \text{if } \theta \in [\frac{1}{3}, 1] \end{cases}$$

Pointwise optimization prescribes setting $k(\theta) = 1$ if $VS(\theta) \geq 0$ and $k(\theta) = 0$ otherwise. In this problem this would result in allocation $k(\theta) = \mathbb{I}\left\{\theta \geq \frac{2}{3}\right\}$, which is not weakly decreasing as monotonicity requires. Conversely, it is weakly increasing. The decreasing function that is "closest" to some increasing function is a constant function. Therefore, let us look at constant allocations $k(\theta) = \bar{k}$ instead. By linearity of buyer's payoff in $\bar{k}$ in that case, one of $\bar{k} \in \{0, 1\}$ must be optimal (and $\bar{k} \in (0, 1)$ cannot be optimal).

Allocation rule corresponding to $\bar{k} = 0$ prescribes no trade w.p. 1, hence buyer's payoff is zero. Allocation rule $\bar{k} = 1$ prescribes trade w.p. 1 at a constant price, hence this price must be $\bar{t} = \max \theta$ for all seller types to be willing to sell. Buyer's expected payoff is then equal to

$$\mathbb{E}\left[v(\theta) - 1\right] = \int_0^{\frac{1}{3}} \frac{3\theta}{2} d\theta + \int_{\frac{1}{3}}^1 \left[\frac{5\theta}{2} - \frac{1}{3}\right] d\theta - 1 = \frac{35}{36} - 1 < 0,$$

hence no trade ($k(\theta) = 0$ for all θ) is optimal.

5. In this case we have

$$VS(\theta) = \begin{cases} \frac{\theta}{2} & \text{if } \theta \in [0, \frac{1}{3}] \\ -\frac{\theta}{2} + \frac{1}{3} & \text{if } \theta \in [\frac{1}{3}, 1] \end{cases}$$

Pointwise maximization yields $k(\theta) = \mathbb{I}\left\{\theta \leq \frac{2}{3}\right\}$. Monotonicity is not violated (the allocation is weakly decreasing in θ), hence this allocation is implementable and, so, optimal.

NOTE: the answers to parts 4 and 5 above do not constitute a full derivation of the optimal mechanism, since they do not handle the buyer's choice of $U_S(1|1)$ and the corresponding discussion of the seller's IR constraints. The problem does not ask for this (and only inquires about the *optimal allocation rule*), but if you were asked instead to derive the full *optimal mechanism*, these extra steps would be expected.

Problem 2: Uber Optimal Algorithm

Suppose you are the head economist of Uber and you are designing the economic side of the matching algorithm. Your goal is to pay the drivers as little as possible, while also ensuring their participation. In particular, consider the following situation: some consumer has placed a fare (a ride order) in the app, and there is a driver available in the vicinity. You (the app) are then effectively bargaining with the driver for how little money they are willing to accept in order to complete this fare.

Suppose the app charges the consumer some amount w for the ride. This w is fixed and commonly known by all players, including the driver and the designer. The driver values their time at $\theta \sim U[0, 1]$, which is their private information. You are designing a direct revelation mechanism $\{k(\theta), p(\theta)\}_{\theta \in [0,1]}$, which works as follows:

(i) the driver reports θ to the app;

(ii) the app offers the fare to the driver with probability $k(\theta) \in [0, 1]$;[2]

(iii) the driver can accept or reject the fare;

(iv) if the driver accepted and completed the fare, they receive payment $p(\theta)$;

(v) if the driver declined or was not offered the fare, they receive utility θ.

Therefore, the driver's expected utility from receiving with probability k a fare that pays p is

$$u(k, p, \theta) = k \cdot p + (1 - k) \cdot \theta,$$

[2] Think that if the fare is not offered to the driver, then the consumer sees an empty screen, as if there were no available drivers in their area.

and their outside option (from rejecting the fare or the whole mechanism) is $\underline{U}(\theta) = \theta$.

Your task is to devise an optimal mechanism (k, p) that maximizes the firm's expected revenue $\mathbb{E}_\theta[k(\theta) \cdot (w - p(\theta))]$ subject to the driver's standard IC constraint and interim IR constraint $u(k, p, \theta) \geq \underline{U}(\theta)$. To derive this mechanism, follow the steps below.

1. Show that in any IC mechanism, $k(\theta)$ must be weakly decreasing.
2. Show that in any IC mechanism, the following holds for all θ (where $U(\theta) \equiv k(\theta)p(\theta) + (1 - k(\theta))\theta$):
$$U(\theta) = U(1) - \int_\theta^1 (1 - k(s))ds. \tag{1}$$
3. Show that in any IC mechanism, the firms' expected profit (from this driver) can be expressed in the following way as a function of the allocation rule k and $U(1)$:
$$\mathbb{E}_\theta[k(\theta) \cdot (w - p(\theta))] = \int_0^1 [2\theta + k(\theta) \cdot (w - 2\theta)]\, d\theta - U(1). \tag{2}$$
4. Find the allocation rule k that maximizes the expected profit (2). Does it satisfy the monotonicity requirement from part 1? If not, what is the optimal monotone allocation rule?
5. Argue (formally if you can) why type $\theta = 1$ is the one for whom the IR constraint will be the most binding. I.e., show that if IR holds for $\theta = 1$ then it also holds for all other $\theta \in [0, 1)$.
6. Suppose $w = 1$. Derive the payment rule $p(\theta)$ that supports the optimal [monotone] allocation rule.
7. Suppose $w = 2$. Derive the payment rule $p(\theta)$ that supports the optimal [monotone] allocation rule.
8. Are the mechanisms you obtained in the two previous questions ex post IR for the drivers? I.e., will the driver always accept any fare that is offered to them?

Solution

1. The argument fully mirrors the argument from class and homeworks. Consider two types $\theta > \theta'$. The IC condition for θ to not be willing to report θ' is:
$$\begin{aligned} U(\theta) \equiv k(\theta)p(\theta) + (1 - k(\theta))\theta &\geq k(\theta')p(\theta') + (1 - k(\theta'))\theta \\ &= U(\theta') + (1 - k(\theta'))(\theta - \theta'). \end{aligned}$$
Rearranging and dividing both sides by $(\theta - \theta')$ yields
$$\frac{U(\theta) - U(\theta')}{\theta - \theta'} \geq 1 - k(\theta').$$
By doing the same manipulations for the IC condition for type θ' to not be willing to report type θ, we obtain
$$1 - k(\theta) \geq \frac{U(\theta) - U(\theta')}{\theta - \theta'} \geq 1 - k(\theta'), \tag{3}$$
hence $k(\theta) \leq k(\theta')$.
2. Taking the limit as $\theta' \to \theta$, (3) becomes $\frac{dU(\theta)}{d\theta} = 1 - k(\theta)$ (for almost all θ). Invoking the fundamental

theorem of calculus, we can say that for any θ then,

$$U(1) - U(\theta) = \int_\theta^1 \frac{dU(s)}{ds} ds = \int_\theta^1 (1 - k(s)) ds. \tag{4}$$

Rearranging this expression yields the result.

3. Using (4) and the definition of $U(\theta)$, we can express the average payment to type θ as

$$k(\theta)p(\theta) = -(1 - k(\theta))\theta - \int_\theta^1 (1 - k(s)) ds + U(1), \tag{5}$$

hence the firm's expected profit from type θ is

$$k(\theta) \cdot (w - p(\theta)) = wk(\theta) + (1 - k(\theta))\theta + \int_\theta^1 (1 - k(s)) ds - U(1).$$

Taking the expectation over types θ, we get the expression for the expected profit:

$$\mathbb{E}_\theta[k(\theta) \cdot (w - p(\theta))] = \int_0^1 \left[wk(\theta) + (1 - k(\theta))\theta + \int_\theta^1 (1 - k(s)) ds \right] d\Phi(\theta) - U(1).$$

Since $\theta \sim U[0, 1]$, we have that $\Phi(\theta) = \theta$ for $\theta \in [0, 1]$. Using integration by parts to eliminate the inner integral and then simplifying the resulting expression, we get

$$\begin{aligned}
\mathbb{E}_\theta[k(\theta) \cdot (w - p(\theta))] &= \int_0^1 [wk(\theta) + (1 - k(\theta))\theta + \theta(1 - k(\theta))] \, d\theta - U(1) \\
&= \int_0^1 [2\theta + k(\theta) \cdot (w - 2\theta)] \, d\theta - U(1).
\end{aligned}$$

4. Maximizing the expression under the integral for any θ, we get that $k(\theta) = \mathbb{I}\left\{\theta \leq \frac{w}{2}\right\}$. This allocation rule is indeed weakly decreasing.

5. Recall $\underline{U}(\theta) = \theta$. From (4) (equivalently, (2)) and the fact that $k(\theta) \geq 0$, we can infer that for all θ,

$$\begin{aligned}
&U(1) - U(\theta) \leq 1 - \theta \\
\Leftrightarrow &U(1) - 1 \leq U(\theta) - \theta.
\end{aligned}$$

Hence if $U(1) \geq 1 \iff U(1) - 1 \geq 0$, then $U(\theta) - \theta \geq 0 \iff U(\theta) \geq \theta$.

6. Transfers are pinned down by k and $U(1)$ by the envelope representation of payoffs (2). We have k from part 4. From part 5 we know that $U(1) = 1$ is both necessary and sufficient for all IR constraints to hold and for at least one of them to bind. Plugging the both of these into (5), we get that for $\theta \leq \frac{w}{2}$,

$$\begin{aligned}
k(\theta)p(\theta) &= -(1 - k(\theta))\theta - \int_\theta^1 (1 - k(s)) ds + U(1) \\
\Rightarrow \text{ if } k(\theta) = 1, \; p(\theta) &= -\int_\theta^1 (1 - k(s)) ds + 1 \\
&= -\int_\theta^{w/2} 0 ds - \int_{w/2}^1 1 ds + 1 \\
&= 1 - \left(1 - \frac{w}{2}\right) = \frac{w}{2}.
\end{aligned}$$

Hence if $w = 1$, any driver that receives the fare ($\theta < \frac{w}{2}$) is paid $p(\theta) = 1/2$. Drivers who report

$\theta > w/2$ are not offered a fare and are paid nothing.[3]

7. Using the results from the previous parts, if $w = 2$ then the optimal mechanism is $k(\theta) = 1$ for all θ and $p(\theta) = 1$, i.e., the driver receives this offer and is compensated handsomely regardless of their report.
8. Yes. If $\theta < w/2$, the driver is offered a fare and a reward $p(\theta) = w/2$, so accepting the fare for $p(\theta)$ is better than rejecting and receiving $\underline{U}(\theta) = \theta$.

Problem 3: Divine intervention 2

Consider the "Divine intervention" problem (L5 exercises, problem 4).

Suppose now instead that the gods are not benevolent, but try to minimize the amount of favor they owe to mortals – i.e., they maximize $\mathbb{E}[t_H + t_G]$. That said, they still like to see a good battle, so ensuring that neither player decides to flee takes priority.

Find the optimal BIC mechanism that maximizes the expected "revenue" $\mathbb{E}_\theta[t_H(\theta) + t_G(\theta)]$ subject to the players' IR constraints.

Solution

Invoke BIC ERP (we can use it since the problem fits the linear setting):

$$\bar{U}_i(\theta_i) = \bar{U}_i(0) + \int_0^{\theta_i} \mathbb{E}_{\theta_{-i}}[k_i(s, \theta_{-i})]ds \tag{6}$$

and substitute the definition of the expected utility $\bar{U}_i(\theta_i)$ to get

$$\begin{aligned}
\mathbb{E}_{\theta_{-i}}[t_i(\theta_i, \theta_{-i})] &= \theta_i \mathbb{E}_{\theta_{-i}}[k_i(\theta_i, \theta_{-i})] - \bar{U}_i(0) - \int_0^{\theta_i} \mathbb{E}_{\theta_{-i}}[k_i(s, \theta_{-i})]ds && (7)\\
&= \int_0^1 \theta_i k_i(\theta_i, \theta_{-i})\phi(\theta_{-i})d\theta_{-i} - \int_0^1 \left(\int_0^{\theta_i} k_i(s, \theta_{-i})ds\right)\phi(\theta_{-i})d\theta_{-i} - \bar{U}_i(0). && (8)
\end{aligned}$$

The second line rewrites the expectations as integrals. Taking the expectation of both sides over θ_i as well and recalling that pdfs are $\phi(\theta_i) = 1$ for $U[0, 1]$ distribution, we get

$$\begin{aligned}
\mathbb{E}_\theta[t_i(\theta_i, \theta_{-i})] &= \int_0^1 \int_0^1 \theta_i k_i(\theta_i, \theta_{-i})d\theta_{-i}d\theta_i - \int_0^1 \int_0^1 \int_0^{\theta_i} k_i(s, \theta_{-i})dsd\theta_i d\theta_{-i} - \bar{U}_i(0) && (9)\\
&= \int_0^1 \int_0^1 \theta_i k_i(\theta_i, \theta_{-i})d\theta_{-i}d\theta_i - \int_0^1 \int_0^1 (1-\theta_i)k_i(\theta_i, \theta_{-i})d\theta_i d\theta_{-i} - \bar{U}_i(0) && (10)\\
&= \int_0^1 \int_0^1 (\theta_i k_i(\theta) - (1-\theta_i)k_i(\theta))\, d\theta_i d\theta_{-i} - \bar{U}_i(0) && (11)\\
&= \int_0^1 \int_0^1 ((2\theta_i - 1)k_i(\theta))\, d\theta_i d\theta_{-i} - \bar{U}_i(0). && (12)
\end{aligned}$$

In the above, from (9) to (10) we used integration by parts as in class, then combined the two integrals to get (11) and simplified the resulting expression in (12). The total expected "revenue" (negative of the favor

[3] The exact value of $p(\theta)$ is indeterminate in this case, since p is defined as payment *conditional* on completing the fare.

owed) from both players is then given by

$$\mathbb{E}_\theta\left[t_H(\theta)+t_G(\theta)\right]=\int_0^1\int_0^1\left[(2\theta_H-1)k_H(\theta)+(2\theta_G-1)k_G(\theta)\right]d\theta_H d\theta_G-\bar{U}_G(0)-\bar{U}_H(0). \tag{13}$$

The goal is to maximize the expression above over k. To maximize the integral pointwise (θ-by-θ), we would want to set $k_i(\theta)=-1$ if $\theta_i<0.5$ and $k_i(\theta)=1$ if $\theta_i>0.5$. The former is not a problem – in particular, if both $\theta_H,\theta_G<0.5$, then both Horik and Guttorm would die in battle.[4] The latter, however, is not possible whenever both $\theta_H,\theta_G>0.5$, since we have the constraint that $k_H+k_G\le 0$ – both cannot win at the same time. The constrained-optimal allocation that maximizes $(2\theta_H-1)k_H(\theta)+(2\theta_G-1)k_G(\theta)$ s.t. $k_H+k_G\le 0$ would be giving the victory to the person with the higher θ_i when $\theta_H,\theta_G>0.5$. In the end, the allocation rule in the optimal mechanism is given by

$$k(\theta)=\begin{cases}(1,-1) & \text{if } \theta_H>\max\{\theta_G,0.5\};\\(-1,1) & \text{if } \theta_G>\max\{\theta_H,0.5\};\\(-1,-1) & \text{if } 0.5\ge\max\{\theta_H,\theta_G\}.\end{cases}$$

Monotonicity holds: $k_i(\theta_i,\theta_{-i})$ is weakly increasing in θ_i for all θ_{-i}, hence $\mathbb{E}_{\theta_{-i}}[k_i(\theta_i,\theta_{-i})]$ is also weakly increasing in θ_i, hence this allocation rule k is indeed implementable in BNE. In fact, the former statement implies that k is even implementable in dominant strategies.

To calculate transfers, go back to ERP. You can actually use either BIC, or DSIC ERP to obtain transfers that will support k in BNE or in dominant strategies, respectively. Let us take BIC ERP:

$$\begin{aligned}\mathbb{E}_{\theta_{-i}}[t_i(\theta_i,\theta_{-i})]&=\theta_i\mathbb{E}_{\theta_{-i}}[k_i(\theta_i,\theta_{-i})]-\bar{U}_i(0)-\int_0^{\theta_i}\mathbb{E}_{\theta_{-i}}[k_i(s,\theta_{-i})]ds\\&=\begin{cases}\theta_i(-1)-\int_0^{\theta_i}(-1)ds-\bar{U}_i(0) & \text{if } \theta_i\le 0.5;\\\theta_i(2\theta_i-1)-\int_0^{0.5}(-1)ds-\int_{0.5}^{\theta_i}(2s-1)ds-\bar{U}_i(0) & \text{if } \theta_i>0.5;\end{cases}\\&=\begin{cases}-\bar{U}_i(0) & \text{if } \theta_i\le 0.5;\\\theta_i^2+\frac{1}{4}-\bar{U}_i(0) & \text{if } \theta_i>0.5.\end{cases}\end{aligned}$$

This gives us the expected (over θ_{-i}) transfer of i given θ_i, but how do can we get $t_i(\theta_i,\theta_{-i})$? A simple answer is: unless we care about ex post IR or BB, there is no real reason in BIC mechanisms to make t_i contingent on θ_{-i}. I.e., given θ_i, for any θ_{-i} we can just set i's transfer to be equal to[5]

$$t_i(\theta_i)=\mathbb{E}_{\theta_{-i}}[t_i(\theta_i,\theta_{-i})]=\begin{cases}-\bar{U}_i(0) & \text{if } \theta_i\le 0.5;\\\theta_i^2+\frac{1}{4}-\bar{U}_i(0) & \text{if } \theta_i>0.5.\end{cases}$$

We are almost done. The only thing left is to pin down the utilities $\bar{U}_i(0)$, which would also pin down the transfers. According to (13), we want to set them as low as possible, but we also want to satisfy IR. Write

[4] According to legends, this was the actual outcome: "A huge battle was fought which lasted for three days. King Horik I "and the other kings" were killed, as were Guttorm and a great many chiefs". (`https://en.wikipedia.org/wiki/Horik_I\#Downfall`)

[5] An alternative path is, as was mentioned previously, to use DSIC ERP to obtain transfers t that support our k in DS. If you did that, you would get

$$t_i(\theta_i,\theta_{-i})=\begin{cases}-U_i(0,\theta_{-i}) & \text{if } \theta_i\le 0.5;\\2\max\{\theta_{-i},0.5\}-U_i(0,\theta_{-i}) & \text{if } \theta_i>0.5.\end{cases}$$

out players' expected utility:

$$\begin{aligned}\bar{U}_i(\theta_i) &= \theta_i \mathbb{E}_{\theta_{-i}}[k_i(\theta_i, \theta_{-i})] - \mathbb{E}_{\theta_{-i}}[t_i(\theta_i, \theta_{-i})] \\ &= \begin{cases} -\theta_i + \bar{U}_i(0) & \text{if } \theta_i \leq 0.5; \\ \theta_i^2 - \theta_i - \frac{1}{4} + \bar{U}_i(0) & \text{if } \theta_i > 0.5. \end{cases}\end{aligned}$$

It is minimized at $\theta_i = 0.5$, meaning that if the IR constraint holds for type $\theta_i = 0.5$ then it will also hold for all other types.Setting the IR constraint to bind for that type ($\bar{U}_i(0.5) = 0$), we obtain $\bar{U}_i(0) = 0.5$, so in the end, the transfers that support the favor-minimizing allocation k in BNE are given by

$$t_i(\theta_i) = \begin{cases} -0.5 & \text{if } \theta_i \leq 0.5; \\ \theta_i^2 - \frac{1}{4} & \text{if } \theta_i > 0.5. \end{cases}$$