

Exercises for Lecture 6: Optimal mechanisms.

Problem 1: Optimal Procurement Mechanism

Consider an inverted setting to the one discussed in the lecture: a seller now has an item of some privately known quality $\theta \in [0, 1]$. Here θ also equals the seller's valuation of the item, and the buyer's valuation is given by $v(\theta) > \theta$. The *buyer* designs a direct revelation mechanism (k, t) to purchase this item, where $k(\theta)$ is the probability of trade and $t(\theta)$ is the payment from the buyer to the seller.¹ Players' expected utilities given the seller's true type θ and the seller's report $\hat{\theta}$ are then given by

$$\begin{aligned} U_S(\hat{\theta}|\theta) &= -\theta k(\hat{\theta}) + t(\hat{\theta}), \\ U_B(\hat{\theta}|\theta) &= v(\theta)k(\hat{\theta}) - t(\hat{\theta}). \end{aligned}$$

1. Show that for an allocation rule $k(\theta)$ to be IC for the seller, it must be weakly *decreasing* in θ .
2. ERP implies that the payoff of the seller of type θ can be written as $U_S(\theta|\theta) = U_S(1|1) + \int_{\theta}^1 k(s)ds$. Using this expression, show that the buyer's expected utility in any IC DRM (k, t) is given by:

$$\mathbb{E}_{\theta}[U_B(\theta|\theta)] = \int_0^1 k(\theta)VS(\theta)\phi(\theta)d\theta - U_S(1|1),$$

where $VS(\theta) = v(\theta) - \theta - \frac{\Phi(\theta)}{\phi(\theta)}$.

3. Explain each component of $VS(\theta)$.
4. Suppose from now on that $\theta \sim U[0, 1]$. Find the optimal allocation rule when $v(\theta) = \frac{3\theta}{2}$ for $\theta \in [0, \frac{1}{3}]$ and $v(\theta) = \frac{5\theta}{2} - \frac{1}{3}$ for $\theta \in [\frac{1}{3}, 1]$.
5. Find the optimal allocation rule when $v(\theta) = \frac{5\theta}{2}$ for $\theta \in [0, \frac{1}{3}]$ and $v(\theta) = \frac{3\theta}{2} + \frac{1}{3}$ for $\theta \in [\frac{1}{3}, 1]$.

Problem 2: Uber Optimal Algorithm

Suppose you are the head economist of Uber and you are designing the economic side of the matching algorithm. Your goal is to pay the drivers as little as possible, while also ensuring their participation. In particular, consider the following situation: some consumer has placed a fare (a ride order) in the app, and there is a driver available in the vicinity. You (the app) are then effectively bargaining with the driver for how little money they are willing to accept in order to complete this fare.

Suppose the app charges the consumer some amount w for the ride. This w is fixed and commonly known by all players, including the driver and the designer. The driver values their time at $\theta \sim U[0, 1]$, which is their private information. You are designing a direct revelation mechanism $\{k(\theta), p(\theta)\}_{\theta \in [0, 1]}$, which works as follows:

- (i) the driver reports θ to the app;
- (ii) the app offers the fare to the driver with probability $k(\theta) \in [0, 1]$;²
- (iii) the driver can accept or reject the fare;
- (iv) if the driver accepted and completed the fare, they receive payment $p(\theta)$;

¹Note that this formulation of the problem explicitly constrains the mechanism to be ex post budget balance requirement.

²Think that if the fare is not offered to the driver, then the consumer sees an empty screen, as if there were no available drivers in their area.

- (v) if the driver declined or was not offered the fare, they receive utility θ .

Therefore, the driver's expected utility from receiving with probability k a fare that pays p is

$$u(k, p, \theta) = k \cdot p + (1 - k) \cdot \theta,$$

and their outside option (from rejecting the fare or the whole mechanism) is $\underline{U}(\theta) = \theta$.

Your task is to devise an optimal mechanism (k, p) that maximizes the firm's expected revenue $\mathbb{E}_\theta[k(\theta) \cdot (w - p(\theta))]$ subject to the driver's standard IC constraint and interim IR constraint $u(k, p, \theta) \geq \underline{U}(\theta)$. To derive this mechanism, follow the steps below.

1. Show that in any IC mechanism, $k(\theta)$ must be weakly decreasing.
2. Show that in any IC mechanism, the following holds for all θ (where $U(\theta) \equiv k(\theta)p(\theta) + (1 - k(\theta))\theta$):

$$U(\theta) = U(1) - \int_\theta^1 (1 - k(s))ds. \quad (1)$$

3. Show that in any IC mechanism, the firms' expected profit (from this driver) can be expressed in the following way as a function of the allocation rule k and $U(1)$:

$$\mathbb{E}_\theta[k(\theta) \cdot (w - p(\theta))] = \int_0^1 [2\theta + k(\theta) \cdot (w - 2\theta)] d\theta - U(1). \quad (2)$$

4. Find the allocation rule k that maximizes the expected profit (2). Does it satisfy the monotonicity requirement from part 1? If not, what is the optimal monotone allocation rule?
5. Argue (formally if you can) why type $\theta = 1$ is the one for whom the IR constraint will be the most binding. I.e., show that if IR holds for $\theta = 1$ then it also holds for all other $\theta \in [0, 1]$.
6. Suppose $w = 1$. Derive the payment rule $p(\theta)$ that supports the optimal [monotone] allocation rule.
7. Suppose $w = 2$. Derive the payment rule $p(\theta)$ that supports the optimal [monotone] allocation rule.
8. Are the mechanisms you obtained in the two previous questions ex post IR for the drivers? I.e., will the driver always accept any fare that is offered to them?

Problem 3: Divine intervention 2

Consider the "Divine intervention" problem (L5 exercises, problem 4).

Suppose now instead that the gods are not benevolent, but try to minimize the amount of favor they owe to mortals – i.e., they maximize $\mathbb{E}[t_H + t_G]$. That said, they still like to see a good battle, so ensuring that neither player decides to flee takes priority.

Find the optimal BIC mechanism that maximizes the expected "revenue" $\mathbb{E}_\theta[t_H(\theta) + t_G(\theta)]$ subject to the players' IR constraints.