Exercises for Lecture 4: Revenue equivalence

Problem 1: Payoff equivalence in BIC mechanisms

Prove the payoff equivalence result for BIC mechanisms using the analog of the argument we had for DSIC mechanisms (i.e., by showing monotonicity first). Assume that players' types are mutually independent.

Problem 2: Efficient public good provision 2

Consider the public good provision problem from the previous problem set, described as follows.

There is a society of N people. They must collectively decide whether to implement a public project. Let $k \in \{0, 1\}$ denote the outcome of this decision: $k = 1$ if project is implemented, $k = 0$ otherwise. Every person i has some private valuation $\theta_i \in \mathbb{R}$ for the project, positive or negative. Preferences are linear, so i's utility can be written as

$$u_i(x, \theta) = \theta_i k(\theta) - t_i(\theta).$$

Here $x = (k, t)$ stands for some direct mechanism which prescribes outcome $k(\theta) \in \{0, 1\}$ and payment profile $t(\theta)$ given profile of reports θ.

1. Fix an arbitrary i and θ_{-i} and suppose that we want to implement (in dominant strategies) an allocation rule that satisfies $k_i(\theta_i, \theta_{-i}) = \mathbb{I}\{\theta_i \geq \tilde{\theta}_i\}$ for some $\tilde{\theta}_i$ (where $\mathbb{I}\{\cdot\}$ is an indicator function).

   Use the envelope representation of payoffs to derive the (class of) transfer rule(s) for i that DS-implements k_i given θ_{-i}.
2. Suppose the public project has cost $c > 0$. Find the efficient allocation rule $k^*(\theta)$. Use your findings from the previous question to derive the (class of) transfer rule(s) for all players that implements k^* in dominant strategies.
3. Conclude that *only* Groves' transfers can DS-implement k^* in this problem.

Problem 3: Implementation in auctions

Consider the auction environment with two buyers and one item for sale. Suppose that each buyer i has willingness to pay $\theta_i \in [0, 1]$.

1. Draw a square with θ_1 on the horizontal axis and θ_2 on the vertical axis. A point in the square represents a profile (θ_1, θ_2).
2. Draw a downward sloping curve through the box.
3. Draw an upward sloping curve through the box that intersects the downward sloping curve (exactly once.)
4. The region above your downward sloping curve is divided into two subregions by your upward sloping curve. Label the subregion that is above your upward sloping curve with a 2 and label the other subregion with a 1. Label the entire region that is below (and to the left of) your downward sloping curve with a 0.
5. Consider the allocation rule that is defined by your drawing. In the 1 region agent 1 gets the good, in the 2 region agent 2 gets the good and in the 0 region neither agent gets the good. (On the boundary

between regions pick the allocation from one of the neighboring regions.) Find a transfer rule which, when coupled with your allocation rule, forms a DSIC mechanism.

Hint: you will probably need to use the envelope representation of payoffs:

$$U_i(\theta_i, \theta_{-i}) = U_i(\underline{\theta}_i, \theta_{-i}) + \int_{\underline{\theta}_i}^{\theta_i} k_i(s, \theta_{-i}) ds$$

In particular, you can derive transfers that support k using the expression above. But if you did Problem 3 above, you already know that.

6. Is there any DSIC allocation rule that picks alternatives from the set $\{0, 1, 2\}$ that could not be represented by a drawing that follows the instructions given above?
(If not: argue verbally why not. If yes: draw a graph and argue intuitively why the k you drew would be DSIC.)

Problem 4: Payoff equivalence in auctions

Consider an auction for one item and N bidders with valuations $\theta_i \sim i.i.d. U[0, 1]$ and linear preferences. Consider three different auction formats, in which all bidders submit bids simultaneously:

- *First-price sealed bid auction*, in which the highest bidder wins the item and pays their own bid. In such a format with $\theta_i \sim U[0, 1]$, bidder i's equilibrium bidding strategy is $b_i^{FPA} = \frac{N-1}{N}\theta_i$.
- *Second-price sealed bid auction*, in which the highest bidder wins and pays the second-highest bid. In such a format, it is a weakly dominant strategy for bidder i to bid their valuation: $b_i^{SPA} = \theta_i$.
- *All-pay auction*, in which all bidders pay their bids, and the highest bidder wins the item. In such a format with $\theta_i \sim U[0, 1]$, bidder i's equilibrium bidding strategy is $b_i^{APA} = \frac{N-1}{N}\theta_i^N$.

Calculate the bidders' (interim) expected payoffs and the auctioneer's (ex ante) expected revenues under the three formats. Verify that they are the same across the three cases.

Bonus question: verify that the bidding functions given for FPA and APA do indeed constitute an equilibrium.