Midterm assignment

Be prepared that the problems below can be more messy and/or difficult than the problem sets. Group submissions are allowed and encouraged (no more than 5 people per group). If something in the assignment is ambiguous or you think something is incorrect, email me.

Problem 1: (Malevolent) Judicial design

A suspect is in custody, accused of murder. If he goes to trial he will either be convicted or acquitted. If he is convicted he will be sent to prison for life giving him a payoff of -1. If he is acquitted he goes free and has a payoff of 0. The district attorney can offer plea bargains: allowing the defendant to plead guilty in return for a lighter sentence. In particular, for any $r \in (0, 1)$, the DA can offer a reduced sentence which, if accepted, would give the defendant a payoff of $-r$.

The defendant is privately informed about his chances for acquittal at trial: $\theta \in [0, 1]$ is the defendant's privately known probability of acquittal. If the defendant does not enter into a plea bargain with the DA he will go to trial and be convicted with probability $1 - \theta$.

Consider the mechanism design problem where the DA is the principal and the defendant is the agent. A social choice function is a mapping $f : [0, 1] \to \{\text{trial}\} \cup (0, 1)$ where $f(\theta) = \text{trial}$ means that type θ will go to trial and $f(\theta) = r \in (0, 1)$ means that type θ accepts a plea bargain giving him a sentence with payoff $-r$. DA thinks θ has full support on $[0, 1]$.

1. Write down the inequalities that characterize whether some given social choice function f is incentive-compatible for the defendant.
2. What is the set of all incentive-compatible social choice functions? You can proceed in the following steps:
   - Show that in any IC f at most one plea bargain r is available.
   - Show that f must be of cutoff type, with the suspect taking the plea if $\theta < \bar{\theta}$ and going to court otherwise.
   - Find the value of r that makes the cutoff s.c.f. f incentive compatible given some cutoff type $\bar{\theta}$.
   - Combine all of the above to characterize the set of implementable f.

Suppose that the DA wants to maximize the expected length of the defendant's sentence, i.e. to minimize the defendant's expected payoff. (So the DA gets a payoff of 1 for a life sentence and a payoff of r for a reduced sentence which would give the defendant a payoff of $-r$.)

3. Among the incentive-compatible mechanisms you identified, what is the optimal mechanism for the DA?
4. How does your answer change if going to trial imposes additional cost $c \in (0, 1)$ on the DA (but not on the defendant) relative to agreeing on a plea bargain?

Problem 2: Piece of cake

Young siblings Annie and Billy are fighting over a cake of size 1. Their respective valuations are given by $\theta_A \geq 0$ and $\theta_B \geq 0$ per unit of cake respectively and are their private information. Both kids act in pure self-interest. Their Dad decides to employ the VCG mechanism to resolve the fight.[1] However, he

[1] Mom, on the other hand, prefers a Vickrey-Clarke-Groves-Weinersmith mechanism: `https://www.smbc-comics.com/comic/mechanism`.

also has preference for splitting the cake equally among the two kids: his (real) utility function is given by $v_0(k) = -\alpha(k_A - k_B)^2$, where k_i is the share of the cake allocated to kid $i = A, B$.

1. Write down the social welfare function that is maximized by the efficient allocation $k^*(\theta)$. Explain the meaning of the parameter α. Derive $k^*(\theta)$.
2. Derive the VCG transfers and describe the whole mechanism. (If you cannot derive the mechanism for the general case, assume $\theta_i \in [0, 1]$, $\alpha > 1/4$, and derive the mechanism for this special case.)
3. Since the kids are unlikely to have any money, what instrument can Dad use as transfers?

Problem 3: Used Car Auction

Monica is running a used car auction. This week she has two cars for sale: a '85 Ford Mustang and an '87 Pontiac Trans Am, hereinafter denoted as $c \in \{F, P\}$. The auction has attracted N interested bidders $i \in \{1, ..., N\}$, whose valuations are commonly believed to be $\theta_{i,c} \sim$ i.i.d.$U[0, 1]$. In particular, for every i, $\theta_{i,F}$ is independent of $\theta_{i,P}$, since the two cars are quite different and have different age-related issues. However, once a bidder wins one car, they are not interested in bidding for another. Monica's value for retaining either car is $\bar{\theta} \in [0, 1]$ and $2\bar{\theta}$ if she retains both. All players' preferences are Euclidean. Your goal is to help Monica design the auction in such a way as to generate the most revenue.

1. Suppose the cars are auctioned sequentially over two periods $t = 1, 2$, and at $t = 2$ there are only one car $c = P$ and $N - 1$ bidders left. Derive the optimal auction (for $t = 2$) that maximizes Monica's expected revenue. Make sure to describe both the allocation and the payment rules.
2. Calculate buyer i's ex ante expected utility from participating in the auction you derived.
3. Now move on to $t = 1$ and the auction for $c = F$. Suppose that at this point the buyers do not yet know their valuations $\theta_{i,P}$ for the second car (since it has not yet been presented and they did not have a chance to inspect it). Derive the optimal auction for $c = F$ in $t = 1$, assuming that in $t = 2$ the auction for $c = P$ will be run according to the rules you derived in part 1.
4. How do you think the expected revenue R_F from selling $c = F$ in $t = 1$ compares with the expected revenue from selling $c = P$ in $t = 2$? (A convincing intuitive argument suffices.) What implications do your conclusions have for auction design? (I.e., is it optimal to sell the two items sequentially or could a different format yield better results?)