Midterm assignment

Be prepared that the problems below can be more messy and/or difficult than the problem sets. Group submissions are allowed and encouraged (no more than 5 people per group). If something in the assignment is ambiguous or you think something is incorrect, email me.

Problem 1: (Malevolent) Judicial design

A suspect is in custody, accused of murder. If he goes to trial he will either be convicted or acquitted. If he is convicted he will be sent to prison for life giving him a payoff of -1. If he is acquitted he goes free and has a payoff of 0. The district attorney can offer plea bargains: allowing the defendant to plead guilty in return for a lighter sentence. In particular, for any $r \in (0, 1)$, the DA can offer a reduced sentence which, if accepted, would give the defendant a payoff of $-r$.

The defendant is privately informed about his chances for acquittal at trial: $\theta \in [0, 1]$ is the defendant's privately known probability of acquittal. If the defendant does not enter into a plea bargain with the DA he will go to trial and be convicted with probability $1 - \theta$.

Consider the mechanism design problem where the DA is the principal and the defendant is the agent. A social choice function is a mapping $f : [0, 1] \to \{\text{trial}\} \cup (0, 1)$ where $f(\theta) = \text{trial}$ means that type θ will go to trial and $f(\theta) = r \in (0, 1)$ means that type θ accepts a plea bargain giving him a sentence with payoff $-r$. DA thinks θ has full support on $[0, 1]$.

1. Write down the inequalities that characterize whether some given social choice function f is incentive-compatible for the defendant.
2. What is the set of all incentive-compatible social choice functions? You can proceed in the following steps:
   - Show that in any IC f at most one plea bargain r is available.
   - Show that f must be of cutoff type, with the suspect taking the plea if $\theta < \bar{\theta}$ and going to court otherwise.
   - Find the value of r that makes the cutoff s.c.f. f incentive compatible given some cutoff type $\bar{\theta}$.
   - Combine all of the above to characterize the set of implementable f.

Suppose that the DA wants to maximize the expected length of the defendant's sentence, i.e. to minimize the defendant's expected payoff. (So the DA gets a payoff of 1 for a life sentence and a payoff of r for a reduced sentence which would give the defendant a payoff of $-r$.)

3. Among the incentive-compatible mechanisms you identified, what is the optimal mechanism for the DA?
4. How does your answer change if going to trial imposes additional cost $c \in (0, 1)$ on the DA (but not on the defendant) relative to agreeing on a plea bargain?

Solution

1. By going to trial a defendant of type θ receives (expected) utility of $-(1 - \theta)$, while from accepting a plea bargain his utility is $-r$. Fix some s.c.f. $f(\theta)$. Let Θ_p be the set of types who are offered a plea bargain $f(\theta) = r(\theta)$, and Θ_t be the set of types who are meant to go to trial: $f(\theta) = \text{trial}$

$(\Theta_t \cup \Theta_p = [0,1])$. Then the IC constraints are given by:

$$\begin{aligned} \text{for all } \theta \in \Theta_p: \quad & -r(\theta) \geq -r(\theta') \text{ for all } \theta' \in \Theta_p \\ & \text{and} - r(\theta) \geq -(1-\theta); \\ \text{for all } \theta \in \Theta_t: \quad & -(1-\theta) \geq -r(\theta') \text{ for all } \theta' \in \Theta_p. \end{aligned}$$

2. We will characterize the set of IC social choice functions by a series of claims.

claim 1 $f(\bullet)$ has at most one value on the real line.

Proof: if $f(\theta_1) < f(\theta_2)$ $\theta_1, \theta_2 \in [0,1]$ then a defendant of type θ_2 gains higher utility by declaring θ_1 (as $-f(\theta_1) > -f(\theta_2)$. This implies the mechanism is not IC for θ_2.

claim 2 $f(\bullet)$ has a cutoff at some $\bar{\theta}$. i.e. $f(\theta) = \begin{cases} r & \text{if } \theta < \bar{\theta} \\ T & \text{if } \theta \geq \bar{\theta} \end{cases}$ (value at $\bar{\theta}$ is not unique)

Proof: assume $\theta' > \theta, f(\theta) = T, f(\theta') = r$. By IC for θ we know that $-r \leq -(1-\theta)$. However as $-(1-\theta') > -(1-\theta)$ this implies that $-(1-\theta') > -r$ and we don't have IC for θ'.

claim 3 $u(-f(\bar{\theta}), \bar{\theta}) \geq -(1-\bar{\theta})$

This follows immediately from IC for type $\bar{\theta}$.

claim 4 $r = 1-\bar{\theta}$

Proof: $r \leq 1-\bar{\theta}$ follows directly from the last claim, while $r \geq 1-\bar{\theta}$ follows from IC of type $\bar{\theta}+\epsilon$. If type $\bar{\theta}$ were strictly better of by accepting the plea bargain, by continuity and monotonicity of benefit of trial, type $\bar{\theta}+\epsilon$ would also strictly prefer the plea bargain contradicting IC for that type.

These four claims imply that for any $(r, \bar{\theta})$ s.t. $r = 1-\bar{\theta}$ the social choice function

$$f(\theta) = \begin{cases} r & \text{if } \theta < \bar{\theta} \\ T & \text{if } \theta \geq \bar{\theta} \end{cases}$$

is incentive compatible.

3. The implementable social choice functions must look as follows, for some $\bar{\theta} \in [0,1]$ (see the original problem):

$$f(\theta) = \begin{cases} r & \text{if } \theta < \bar{\theta} \\ T & \text{if } \theta \geq \bar{\theta} \end{cases},$$

where plea bargain $r = 1-\bar{\theta}$ is offered, and the defendant can choose between that and going to trial. The DA gets r from a plea deal and $1-\theta$ if the defendant goes to trial, so the DA's expected payoff from any such $f(\theta)$ is:

$$\begin{aligned} \Phi(\bar{\theta}) \cdot (1-\bar{\theta}) + \int_{\bar{\theta}}^{1} (1-\theta-c) d\Phi(\theta) &= \bar{\theta}(1-\bar{\theta}) + \frac{1-2\bar{\theta}+\bar{\theta}^2}{2} - c(1-\bar{\theta}) \\ &= (1-\bar{\theta}) \left(\frac{1+\bar{\theta}}{2} - c \right) \end{aligned} \tag{1}$$

We could maximize (1) over $\bar{\theta}$ directly. However, for $c = 0$ there is a more direct solution. Any defendant who takes the the plea bargain gets a lower sentence from the plea bargain than what they would have gotten from trial. Thus, it is clearly optimal for the DA not to offer any plea bargains (except perhaps to $\theta = 0$, which will accept a plea bargain of 1), i.e., setting $\bar{\theta} = 0$ is optimal.

4. When going to trial imposes cost $c > 0$ on the DA, maximizing (1) w.r.t. $\bar{\theta}$ yields the optimal threshold $\bar{\theta} = c$. So in the optimal mechanism, a plea deal $r = 1 - c$ is offered, sufficiently innocent types $\theta \in [0, c]$ take it, and types $\theta \in (c, 1]$ prefer to go to trial.

Problem 2: Piece of cake

Young siblings Annie and Billy are fighting over a cake of size 1. Their respective valuations are given by $\theta_A \geq 0$ and $\theta_B \geq 0$ per unit of cake respectively and are their private information. Both kids act in pure self-interest. Their Dad decides to employ the VCG mechanism to resolve the fight.[1] However, he also has preference for splitting the cake equally among the two kids: his (real) utility function is given by $v_0(k) = -\alpha(k_A - k_B)^2$, where k_i is the share of the cake allocated to kid $i = A, B$.

1. Write down the social welfare function that is maximized by the efficient allocation $k^*(\theta)$. Explain the meaning of the parameter α. Derive $k^*(\theta)$.
2. Derive the VCG transfers and describe the whole mechanism. (If you cannot derive the mechanism for the general case, assume $\theta_i \in [0, 1]$, $\alpha > 1/4$, and derive the mechanism for this special case.)
3. Since the kids are unlikely to have any money, what instrument can Dad use as transfers?

Solution

1. The kids' real utilities are standard Euclidean, $v_i(k, \theta) = \theta_i k_i$, so the social welfare is given by

$$w(k, \theta) = \sum_{i \in \{0, A, B\}} v_i(k, \theta) = -\alpha(k_A - k_B)^2 + \theta_A k_A + \theta_B k_B.$$

Since this is effectively Dad's objective function as a designer (as opposed to v_0), α describes the weight he puts on equity relative to the kids' utilities. The efficient allocation that maximizes $w(k, \theta)$ subject to the constraint $k_A + k_B \leq 1$ is given by $k^*(\theta) = (k^*_A(\theta), k^*_B(\theta))$ with $k^*_A(\theta) = \min\left\{\max\left\{\frac{1}{2} + \frac{\theta_A - \theta_B}{8\alpha}, 0\right\}, 1\right\}$ and $k^*_B(\theta) = 1 - k^*_A(\theta)$.

2. First we need to calculate the efficient-excluding-i allocations for $i = A, B$:

$$\begin{aligned} k_i^{-i}(\theta) &= \arg\max_k \left\{-\alpha(k_j - k_i)^2 + \theta_j k_j\right\} \\ &= \min\left\{\max\left\{\frac{1}{2} - \frac{\theta_j}{8\alpha}, 0\right\}, 1\right\}, \end{aligned}$$

and $k_j^{-i}(\theta) = 1 - k_i^{-i}(\theta)$. Note that in this calculation, we only ignore i's utility, but k_i still enters Dad's utility, which is included and favors equity. Hence, if α is large enough, k_i^{-i} will be positive.

[1] Mom, on the other hand, prefers a Vickrey-Clarke-Groves-Weinersmith mechanism: `https://www.smbc-comics.com/comic/mechanism`.

Then applying the standard expression for the VCG transfers:

$$
\begin{aligned}
t_i^{VCG}(\theta) &= -\left(\sum_{j\neq i} v_j(k^*(\theta_i,\theta_{-i}),\theta_j)\right) + \sum_{j\neq i} v_j(k^*(\theta_{-i}),\theta_j) \\
&= -\left(-\alpha\left(k_j^*(\theta)-k_i^*(\theta)\right)^2 + \theta_j k_j^*(\theta)\right) + \left(-\alpha\left(k_j^{-i}(\theta)-k_i^{-i}(\theta)\right)^2 + \theta_j k_j^{-i}(\theta)\right) \\
&= \begin{cases}
-(-\alpha+\theta_j)+(-\alpha+\theta_j) & \text{if } \theta_j \geq 4\alpha+\theta_i; \\
-(-\alpha(\frac{\theta_j-\theta_i}{4\alpha})^2+\theta_j(\frac{1}{2}+\frac{\theta_j-\theta_i}{8\alpha}))+(-\alpha+\theta_j) & \text{if } \theta_j \in [\max\{4\alpha,-4\alpha+\theta_i\},4\alpha+\theta_i]; \\
-(-\alpha)+(-\alpha+\theta_j) & \text{if } \theta_j \in [4\alpha,-4\alpha+\theta_i]; \\
-(-\alpha(\frac{\theta_j-\theta_i}{4\alpha})^2+\theta_j(\frac{1}{2}+\frac{\theta_j-\theta_i}{8\alpha}))+(-\alpha(\frac{\theta_j}{4\alpha})^2+\theta_j(\frac{1}{2}+\frac{\theta_j}{8\alpha})) & \text{if } \theta_j \in [-4\alpha+\theta_i,4\alpha]; \\
-(-\alpha)+(-\alpha(\frac{\theta_j}{4\alpha})^2+\theta_j(\frac{1}{2}+\frac{\theta_j}{8\alpha})) & \text{if } \theta_j \leq \min\{4\alpha,-4\alpha+\theta_i\};
\end{cases} \\
&= \begin{cases}
0 & \text{if } \theta_j \geq 4\alpha+\theta_i; \\
\alpha\left[\left(\frac{\theta_i}{4\alpha}\right)^2-\left(\frac{\theta_j}{4\alpha}-1\right)^2\right] & \text{if } \theta_j \in [\max\{4\alpha,-4\alpha+\theta_i\},4\alpha+\theta_i]; \\
\theta_j & \text{if } \theta_j \in [4\alpha,-4\alpha+\theta_i]; \\
\alpha\left(\frac{\theta_i}{4\alpha}\right)^2 & \text{if } \theta_j \in [-4\alpha+\theta_i,4\alpha]; \\
\alpha\left(\frac{\theta_j}{4\alpha}+1\right)^2 & \text{if } \theta_j \leq \min\{4\alpha,-4\alpha+\theta_i\}.
\end{cases}
\end{aligned}
$$

Combining this with the allocation rule $k^*(\theta)$, we can conclude that the VCG mechanism looks as given in Table 1 and Figure 1. Depending on the parameters of the problem, some of the regions may be empty. E.g., if $\theta_i \in [0,1]$ then for $\alpha \in [1/8, 1/4]$ regions $R2$ and $R5$ disappear, whereas for $\alpha > 1/4$ only region $R6$ remains.

	$k_A^*(\theta)$	$k_B^*(\theta)$	$t_A^{VCG}(\theta)$	$t_B^{VCG}(\theta)$
$\theta \in R1$	0	1	0	$\alpha\left(\frac{\theta_A}{4\alpha}+1\right)^2$
$\theta \in R2$	0	1	0	θ_A
$\theta \in R3$	$\frac{1}{2}+\frac{\theta_A-\theta_B}{8\alpha}$	$\frac{1}{2}-\frac{\theta_A-\theta_B}{8\alpha}$	$\alpha\left[\left(\frac{\theta_A}{4\alpha}\right)^2-\left(\frac{\theta_B}{4\alpha}-1\right)^2\right]$	$\alpha\left(\frac{\theta_B}{4\alpha}\right)^2$
$\theta \in R4$	$\frac{1}{2}+\frac{\theta_A-\theta_B}{8\alpha}$	$\frac{1}{2}-\frac{\theta_A-\theta_B}{8\alpha}$	$\alpha\left[\left(\frac{\theta_A}{4\alpha}\right)^2-\left(\frac{\theta_B}{4\alpha}-1\right)^2\right]$	$\alpha\left[\left(\frac{\theta_B}{4\alpha}\right)^2-\left(\frac{\theta_A}{4\alpha}-1\right)^2\right]$
$\theta \in R5$	1	0	θ_B	0
$\theta \in R6$	$\frac{1}{2}+\frac{\theta_A-\theta_B}{8\alpha}$	$\frac{1}{2}-\frac{\theta_A-\theta_B}{8\alpha}$	$\alpha\left(\frac{\theta_A}{4\alpha}\right)^2$	$\alpha\left(\frac{\theta_B}{4\alpha}\right)^2$
$\theta \in R7$	$\frac{1}{2}+\frac{\theta_A-\theta_B}{8\alpha}$	$\frac{1}{2}-\frac{\theta_A-\theta_B}{8\alpha}$	$\alpha\left(\frac{\theta_A}{4\alpha}\right)^2$	$\alpha\left[\left(\frac{\theta_B}{4\alpha}\right)^2-\left(\frac{\theta_A}{4\alpha}-1\right)^2\right]$
$\theta \in R8$	1	0	$\alpha\left(\frac{\theta_B}{4\alpha}+1\right)^2$	0

Table 1: The VCG mechanism for cake sharing, see Figure 1 for type regions.

3. Within the monetary realm, Dad can withhold kids' future allowance, which should be similar to requiring a payment. Alternatively, methods of payment can include cutting down on the kids' screen time (on a smartphone, tv, Nintendo Switch™, etc), bedtime, curfew time, or similar. Symmetrically, transfers *to* the kids can be implemented by increasing their respective time allowance.

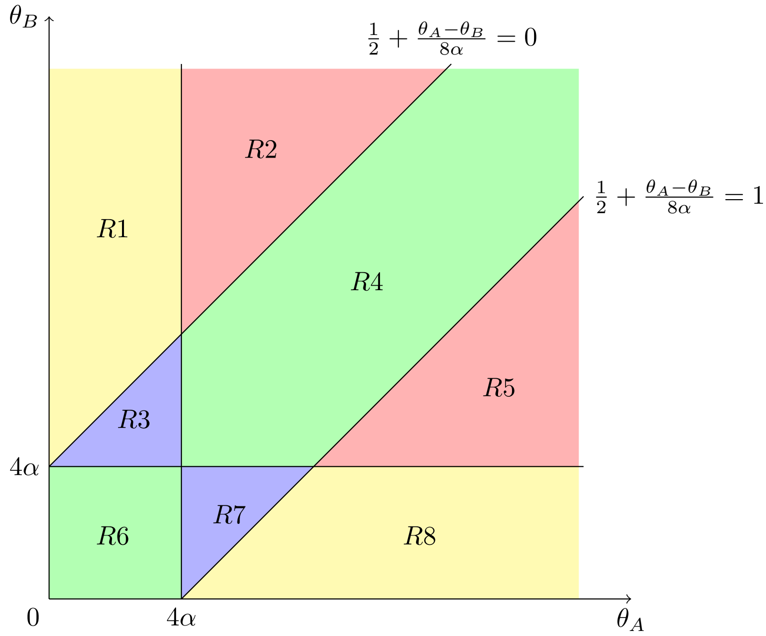

Figure 1: regions of types for the cake problem.

Problem 3: Used Car Auction

Monica is running a used car auction. This week she has two cars for sale: a '85 Ford Mustang and an '87 Pontiac Trans Am, hereinafter denoted as $c \in \{F, P\}$. The auction has attracted N interested bidders $i \in \{1, ..., N\}$, whose valuations are commonly believed to be $\theta_{i,c} \sim$ i.i.d.$U[0, 1]$. In particular, for every i, $\theta_{i,F}$ is independent of $\theta_{i,P}$, since the two cars are quite different and have different age-related issues. However, once a bidder wins one car, they are not interested in bidding for another. Monica's value for retaining either car is $\bar{\theta} \in [0, 1]$ and $2\bar{\theta}$ if she retains both. All players' preferences are Euclidean. Your goal is to help Monica design the auction in such a way as to generate the most revenue.

1. Suppose the cars are auctioned sequentially over two periods $t = 1, 2$, and at $t = 2$ there are only one car $c = P$ and $N - 1$ bidders left. Derive the optimal auction (for $t = 2$) that maximizes Monica's expected revenue. Make sure to describe both the allocation and the payment rules.
2. Calculate buyer i's ex ante expected utility from participating in the auction you derived.
3. Now move on to $t = 1$ and the auction for $c = F$. Suppose that at this point the buyers do not yet know their valuations $\theta_{i,P}$ for the second car (since it has not yet been presented and they did not have a chance to inspect it). Derive the optimal auction for $c = F$ in $t = 1$, assuming that in $t = 2$ the auction for $c = P$ will be run according to the rules you derived in part 1.
4. How do you think the expected revenue R_F from selling $c = F$ in $t = 1$ compares with the expected revenue from selling $c = P$ in $t = 2$? (A convincing intuitive argument suffices.) What implications do your conclusions have for auction design? (I.e., is it optimal to sell the two items sequentially or could a different format yield better results?)

Solution

1. Following the slides for the optimal auctions, we get that Monica's expected profit is given by

$$\begin{aligned}\mathbb{E}U_{M,2} &= \mathbb{E}_\theta\left[\sum_{i=1}^{N-1} t_{i,2}(\theta) - k_{i,2}\bar{\theta}\right] \\ &= \mathbb{E}_\theta\left[\sum_{i=1}^{N-1} \left(k_{i,2}(\theta)VS_{i,2}(\theta) - U_{i,2}(0,\theta_{-i,P})\right)\right], \\ \text{where } VS_{i,2}(\theta) &= \theta_{i,P} - \bar{\theta} - \frac{1-\Phi_i(\theta_{i,P})}{\phi_i(\theta_{i,P})} \\ &= 2\theta_{i,P} - (1+\bar{\theta}).\end{aligned} \tag{2}$$

Note that it is convenient to incorporate $\bar{\theta}$ directly in the objective function as a loss if a trade takes place. If we do this, we can include it in VS as a part of the real surplus generated from trade, $\theta_{i,P} - \bar{\theta}$.

At $t = 2$, bidders' outside option is zero, hence the minimal $U_{i,2}(0, \theta_{i,P})$ we can set is zero for all $i, \theta_{i,P}$. Further, maximizing $\mathbb{E}U_{M,2}$ over allocation rules k that are feasible ($\sum_{i=1}^{N-1} k_{i,2} \leq 1$), we get

$$k^*_{i,2}(\theta) = \begin{cases} 1 & \text{if } \theta_{i,P} \geq \hat{\theta}_{i,2}, \\ 0 & \text{otherwise,} \end{cases}$$

where $\hat{\theta}_{i,2} = \max\left\{\frac{1+\bar{\theta}}{2}, \max_{j\neq i}\{\theta_{j,P}\}\right\}$ is the minimal winning report for i given others' reports. We can see that this allocation rule is monotone (it needs to be increasing in $\theta_{i,P}$ in this problem) for all $\theta_{-i,P}$, hence it is implementable in dominant strategies. To find the transfers that support it, use the ERP for the bidders' utility:

$$\begin{aligned}U_{i,2}(\theta_{i,P}, \theta_{-i,P}) = \theta_{i,P}k_{i,2}(\theta) - t_{i,2}(\theta) &= U_{i,2}(0,\theta_{-i,P}) + \int_0^{\theta_{i,P}} k_{i,2}(\theta)d\theta_{i,P} \\ &= \max\{\theta_{i,P} - \hat{\theta}_{i,2}, 0\} \\ \Rightarrow t_{i,2}(\theta) &= \begin{cases} \hat{\theta}_{i,2} & \text{if } \theta_{i,P} \geq \hat{\theta}_{i,2} \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

It is trivial to verify that the resulting mechanism is IR for all $i, \theta_{i,P}, \theta_{-i,P}$.

We conclude that the optimal auction at $t = 2$ is a second-price auction with reserve price equal to $\frac{1+\bar{\theta}}{2}$.

2. We have calculated that $U_{i,2}(\theta) = \max\{\theta_{i,P} - \hat{\theta}_{i,2}, 0\}$, hence

$$\begin{aligned}\mathbb{E}_\theta U_{i,2}(\theta_i, \theta_{-i}) &= \int_0^1\int_0^1 \max\{\theta_{i,P} - \hat{\theta}_{i,2}, 0\}d\Phi_i(\theta_{i,P})d\hat{\Phi}_i(\hat{\theta}_{i,2}) \\ &= \int_0^1\left[\int_{\hat{\theta}_{i,2}}^1 (\theta_{i,P} - \hat{\theta}_{i,2})d\theta_{i,P}\right]d\hat{\Phi}_i(\hat{\theta}_{i,2}) \\ &= \int_0^1\left[\frac{(1-\hat{\theta}_{i,2})^2}{2}\right]d\hat{\Phi}_i(\hat{\theta}_{i,2}),\end{aligned}$$

where $\hat{\Phi}_i(\cdot)$ is the cdf of $\hat{\theta}_{i,2}$:

$$\begin{aligned}\hat{\Phi}_i(\hat{\theta}_{i,2}) &= \mathbb{P}\left\{\max\left\{\frac{1+\bar{\theta}}{2}, \max_{j\neq i}\{\theta_{j,P}\}\right\} \leq \hat{\theta}_{i,2}\right\} \\ &= \begin{cases} 0 & \text{if } \hat{\theta}_{i,2} < \frac{1+\bar{\theta}}{2} \\ \mathbb{P}\left\{\max_{j\neq i}\{\theta_{j,P}\} \leq \hat{\theta}_{i,2}\right\} & \text{if } \hat{\theta}_{i,2} \geq \frac{1+\bar{\theta}}{2}\end{cases} = \begin{cases} 0 & \text{if } \hat{\theta}_i < \frac{1+\bar{\theta}}{2} \\ \hat{\theta}_{i,2}^{N-2} & \text{if } \hat{\theta}_i \geq \frac{1+\bar{\theta}}{2}\end{cases}\end{aligned}$$

(recall that $\max_{j\neq i}\{\theta_{j,P}\}$ is a max of $N-2$ elements, since it is assumed there is a total of $N-1$ bidders at this stage). So we have (for $\hat{\theta}_{i,2} \geq \frac{1-\bar{\theta}}{2}$) that $d\hat{\Phi}_i(\hat{\theta}_{i,2}) = \hat{\phi}_i(\hat{\theta}_{i,2})d\hat{\theta}_{i,2} = (N-2)\hat{\theta}_{i,2}^{N-3}d\hat{\theta}_{i,2}$, and at $\hat{\theta}_{i,2} = \frac{1+\bar{\theta}}{2}$, $\hat{\Phi}_i(\hat{\theta}_{i,2})$ jumps up from zero to $\frac{1+\bar{\theta}}{2}$. Plugging this in, we get

$$\begin{aligned}\mathbb{E}_\theta U_{i,2}(\theta_{i,P}, \theta_{-i,P}) &= \left(\frac{(1-\hat{\theta}_{i,2})^2}{2}\right)\Bigg|_{\hat{\theta}_{i,2}=\frac{1+\bar{\theta}}{2}} \cdot \left[\left(\frac{1+\bar{\theta}}{2}\right)^{N-2} - 0^{N-2}\right] + \int_{\frac{1+\bar{\theta}}{2}}^{1} \frac{(1-\hat{\theta}_{i,2})^2}{2}(N-2)\hat{\theta}_{i,2}^{N-3}d\hat{\theta}_{i,2} \\ &= \frac{1}{N}\left(\frac{1+\bar{\theta}}{2}\right)^{N} - \frac{1}{N-1}\left(\frac{1+\bar{\theta}}{2}\right)^{N-1} + \frac{1}{N(N-1)} \qquad (3)\end{aligned}$$

It can be verified (analytically or graphically) that this function is decreasing in N.

3. The difference between the two periods is that at $t=1$, the bidders' outside option from not participating in the auction or not winning the item is not zero, since they have an option to participate at $t=2$, which yields positive expected utility. At the same time, if a bidder wins F at $t=1$, they forego this value (since, as assumed in the problem, they have no value for a second car, and will not participate at $t=2$). Letting α denote the probability that Ford is sold at $t=1$ to one of the other $N-1$ bidders, (3) implies the outside option is given by

$$\bar{U} = \frac{1-\alpha}{N+1}\left(\frac{1+\bar{\theta}}{2}\right)^{N+1} + \frac{2\alpha-1}{N}\left(\frac{1+\bar{\theta}}{2}\right)^{N} - \frac{\alpha}{N-1}\left(\frac{1+\bar{\theta}}{2}\right)^{N-1} + \frac{1-\alpha}{(N+1)N} + \frac{\alpha}{N(N-1)}.$$

Bidder i's utility function at $t=1$ is then given by $u_{i,1}(x,\theta) = \theta_{i,F}k_{i,1} + \bar{U}(1-k_{i,1}) - t_{i,1}$. All derivations leading to expression (2) still apply in this case, with the virtual suprlus now being

$$VS_{i,1}(\theta) = (\theta_{i,F} - \bar{U}) - \bar{\theta} - \frac{1-\Phi_i(\theta_{i,F})}{\phi_i(\theta_{i,F})} = 2\theta_{i,F} - (1+\bar{U}+\bar{\theta}).$$

The optimal allocation rule is hence given by

$$k^*_{i,1}(\theta) = \begin{cases} 1 & \text{if } \theta_{i,F} \geq \hat{\theta}_{i,1} \\ 0 & \text{otherwise}\end{cases},$$

where $\hat{\theta}_{i,1} = \max\left\{\frac{1+\bar{U}+\bar{\theta}}{2}, \max_{j\neq i}\{\theta_{j,F}\}\right\}$. However, now the lowest we can set $U_{i,1}(0,\theta_{-i,F})$ to is $U_{i,1}(0,\theta_{-i,F}) = \bar{U}$, implying that the transfers are given by

$$\begin{aligned}\theta_{i,F}k_{i,1} + \bar{U}(1-k_{i,1}) - t_{i,1} &= \bar{U} + \max\{\theta_{i,F} - \hat{\theta}_{i,1}, 0\} \\ \Rightarrow t_{i,1}(\theta) &= \begin{cases} \hat{\theta}_{i,1} - \bar{U} & \text{if } \theta_{i,F} \geq \hat{\theta}_{i,1} \\ 0 & \text{otherwise}\end{cases}.\end{aligned}$$

Note that this is not the final solution: $k^*_{i,1}$ and $t_{i,1}$ both depend on $\hat{\theta}_{i,1}$, which depends on $\bar{U}$, which depends on α, which depends on k^*_1, hence we have a closed system. Resolving this system yields the solution.

Remark: the above adopts an intuitive assumption that entry to the second auction is unrestricted (which is plausible in this setting). However, this does not fully exploit the power of dynamic mechanisms. In particular, Monica could restrict access to the $t = 2$ auction to only those agents who participate at $t = 1$. This means the bidders' outside option $\bar{U}_n$ from dropping out of $t = 1$ auction is then given by $\bar{U}_n = 0$, whereas continuing and not winning car F yields $\bar{U}_p = \bar{U}$ as defined above (as long as all agents choose to participate in equilibrium). Asking all players to pay $\bar{U}_p - \bar{U}_n = \bar{U}$ in order to be admitted to the second-period auction could then serve as a free source of extra revenue. I.e., the optimal first-period mechanism if exclusion is possible consists of $k^*_{i,1}(\theta)$ defined above and

$$t_{i,1}(\theta) = \begin{cases} \hat{\theta}_{i,1} & \text{if } \theta_{i,F} \geq \hat{\theta}_{i,1} \\ \bar{U}_p & \text{otherwise} \end{cases}.$$

4. We can see that at $t = 1$, the item is sold less frequently (since the winner's valuation now must be above $\frac{1+\bar{U}+\bar{\theta}}{2}$, as opposed to the $\frac{1+\bar{\theta}}{2}$ cutoff at $t = 2$), and all bidders shade their bids (the winner pays the second-highest valuation minus $\bar{U}$). These two factors suggest that the expected revenue in $t = 1$ will be lower. However, there is another factor, which is that the competition is more intense, since we have N bidders for sure at $t = 1$ and we may have $N - 1$ bidders at $t = 2$. This effect may dominate for small N, leading to higher revenue at $t = 1$.

   Either way, the total revenue from both periods would be lower than if the bidders did not about the existence of both cars from the start. So it might be optimal to announce an auction for $c = F$, sell that via an SPA with reserve $\frac{1+\bar{\theta}}{2}$, and then announce that $c = P$ is also for sale in another similar SPA. However, that may attract fewer interested bidders N to start with, reducing the revenue again. In the end, the solution is not clear-cut without further calculations and assumptions about bidder participation.