

Mechanism Design

3: Optimal (revenue-maximizing) mechanisms

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Done so far

- Introduced basic notions and criteria:
 - s.c.f., mechanism, implementation and implementability (DSIC, BIC),
 - efficiency, individual rationality, budget balance.
- Covered some fundamental results in Mechanism Design:
 - revelation principle (pretty universal),
 - payoff/revenue equivalence (Euclidean model, slightly generalizable),
 - necessary conditions for implementability (weak preference reversal, monotonicity)
- Learned to implement the efficient s.c.f.:
 - DSIC: VCG;
 - BIC: AGV, gVCG.

Today

- Finished with implementing efficient s.c.f.s
- Today will look at revenue maximization.
 - Revenue-maximizing mechanisms called “optimal” in the literature (meaning optimal for the designer), after Myerson's “optimal mechanism”.
- gVCG was optimal in the class of efficient mechanisms. Now we remove the restriction on allocations.

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This slide deck:

- 1 Two types (Monopolistic Screening)
- 2 Interval of types (Optimal Mechanism)
- 3 Many buyers (Optimal Auction)

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Setting 1: one buyer, discrete type

- Starting simple (**Monopolistic Screening** / Second-Degree Price Discrimination).
- Seller-designer can set quantities k and prices t for product, has production costs $c(k) = k^2 (= -v_0(k))$.
 - As usual, designer has no private information. “Informed principal” is a difficult problem.
- There is one buyer with valuation $\theta \in \{L, H\}$, private info. Prior probabilities are $\phi(H) = \phi$, $\phi(L) = 1 - \phi$.
- Buyer’s preferences Euclidean: $u_b(x, \theta) = \theta k - t$
 - Is this a Euclidean model?

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Monopolistic Screening

- As usual, look at DRM $\Gamma = (\Theta, (k, t))$. Notation-wise, let $k_\theta \equiv k(\theta)$ and $t_\theta \equiv t(\theta)$.
- Seller’s problem (contrary to before, we can now choose k in addition to t .)

$$\max_{(k, t)} \{ \phi(t_H - k_H^2) + (1 - \phi)(t_L - k_L^2) \}$$

$$\text{s.t. } (IC_H) : \theta_H k_H - t_H \geq \theta_H k_L - t_L$$

$$(IC_L) : \theta_L k_H - t_H \leq \theta_L k_L - t_L$$

$$(IR_H) : \theta_H k_H - t_H \geq 0$$

$$(IR_L) : \theta_L k_L - t_L \geq 0$$

- 1 show IC_H and IR_L imply IR_H ;
- 2 show IC_H and IC_L imply $k_H \geq k_L$;
- 3 show $k_H \geq k_L$ and binding IC_H imply IC_L ;
- 4 show IC_H and IR_L bind;
- 5 solve for optimal (k, t) .

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Monopolistic Screening: lessons

Lessons:

- 1 offering a menu may be optimal to extract value from buyers;
- 2 explains weird non-linear prices you can often encounter;
- 3 quantity is distorted downward for low type
- 4 high type gets information rent (pays below valuation);
- 5 IR must bind for at least some type.

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Setting 2: one buyer, interval of types

- Designer/seller has one indivisible item for sale. Chooses menu including probability of sale $k(\theta) \in [0, 1]$ and payment $t(\theta)$ given report θ , no costs for simplicity.
 - Nothing changes from when k was quantity, since everyone is risk-neutral.
- Buyer has valuation $\theta \sim \Phi[0, \bar{\theta}]$, private info.
- Buyer's preferences Euclidean: $u_b = \theta k - t$.
- Buyer's outside option yields utility zero: $\underline{U}_b(\theta) = 0$.

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Optimal Mechanism

Solution approach

- 1 Show: if $\theta' \leq \theta''$ then $k(\theta') \leq k(\theta'')$. (use the two ICs)
- 2 Get the envelope representation $U_b(\theta) = U_b(0) + \int_0^\theta k(s)ds$
(BIC and DSIC ERPs yield the same expression, since there is only one player).
- 3 Recall that $U_b(\theta) = \theta k(\theta) - t(\theta)$, so

$$\begin{aligned}\mathbb{E}U_s &= \mathbb{E}_\theta[t(\theta)] = \mathbb{E}_\theta[\theta k(\theta) - U_b(\theta)] \\ &= \mathbb{E}_\theta\left[\theta k(\theta) - \int_0^\theta k(s)ds\right] - U_b(0) \\ &= \int_0^{\bar{\theta}} \theta k(\theta) \phi(\theta) d\theta - \int_0^{\bar{\theta}} \int_0^\theta k(s) \phi(\theta) ds d\theta - U_b(0) \\ &= \int_0^{\bar{\theta}} \theta k(\theta) \phi(\theta) d\theta - \int_0^{\bar{\theta}} (1 - \Phi(\theta)) k(\theta) d\theta - U_b(0) \\ &= \int_0^{\bar{\theta}} k(\theta) \left(\theta - \frac{1 - \Phi(\theta)}{\phi(\theta)}\right) \phi(\theta) d\theta - U_b(0)\end{aligned}$$

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Optimal Mechanism: integration by parts

Integration by parts under the microscope:

$$\begin{aligned}\int_0^{\bar{\theta}} \left(\int_0^{\theta} k(s) ds \right) \phi(\theta) d\theta &= \left[\Phi(\theta) \int_0^{\theta} k(s) ds \right] \Big|_{\theta=0}^{\bar{\theta}} - \int_0^{\bar{\theta}} \Phi(\theta) k(\theta) d\theta \\ &= \int_0^{\bar{\theta}} k(\theta) d\theta - \int_0^{\bar{\theta}} \Phi(\theta) k(\theta) d\theta \\ &= \int_0^{\bar{\theta}} (1 - \Phi(\theta)) k(\theta) d\theta\end{aligned}$$

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Optimal Mechanism: pinning $U_b(0)$

$$\mathbb{E} U_s = \mathbb{E}_{\theta}[t(\theta)] = \int_0^{\bar{\theta}} k(\theta) \left(\theta - \frac{1 - \Phi(\theta)}{\phi(\theta)} \right) \phi(\theta) d\theta - U_b(0)$$

- To choose: allocation rule $k(\theta)$ and $U_b(0)$ (pins transfers).
- What do with $U_b(0)$?
 - Want to minimize since decreases revenue.
 - Gotta be $U_b(0) \geq 0$ to satisfy IR for $\theta = 0$. Other types?
 - Recall $U_b(\theta) = U_b(0) + \int_0^{\theta} k(s) ds$ and $k(\theta) \geq 0$, so $U_b(\theta) \geq U_b(0)$ for all θ ,
 - hence $U_b(0) = \underline{U}_b(\theta) = 0$ is optimal (max revenue, all IR hold, IR binds for $\theta = 0$).

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Optimal Mechanism: optimal k

$$\mathbb{E}U_s = \int_0^{\bar{\theta}} k(\theta) \left(\theta - \frac{1 - \Phi(\theta)}{\phi(\theta)} \right) \phi(\theta) d\theta$$

- What do with k ?
 - Define **virtual surplus** $VS(\theta) := \theta - \frac{1 - \Phi(\theta)}{\phi(\theta)}$.
 - Pointwise maximization: $k(\theta) = \begin{cases} 1 & \text{if } VS(\theta) \geq 0; \\ 0 & \text{if } VS(\theta) < 0; \end{cases}$
 - Remember: $k(\theta)$ is only implementable if it is monotone! Sufficient condition: $VS(\theta)$ increasing in θ .
 - In the end, if $VS(\theta)$ is increasing in θ , the optimal mechanism is given by $k(\theta)$ as above and $t(\theta)$ that can be computed from ERP.

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Optimal Mechanism: Virtual Surplus

- What is virtual surplus?
 - It reflects **information rents** we have to pay to high types to incentivize them to reveal type honestly.
- Sufficient for increasing $VS(\theta)$ is **increasing hazard rate** $\frac{\phi(\theta)}{1 - \Phi(\theta)}$.
The assumption we usually live with; need suitable distribution $\Phi(\theta)$.
- What do if “unlucky” and $\Phi(\theta)$ is such that VS is sometimes decreasing?
 - “**Ironing**”: find monotone $k(\theta)$ that is “closest” to the unconstrained optimum.
 - E.g. if VS is globally decreasing then some constant k is optimal.
 - There is a kind of a general approach to this, but it’s difficult, see Kleiner, Moldovanu, and Strack [2021].

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Optimal Mechanism: non-linear preferences

- Note that linear preferences $v(k, \theta) = \theta k$ are not necessary for any of this.
- With general v you will not get a nice decomposition $k \cdot VS$ in the integral.
- But you can still obtain something like

$$\int_0^{\bar{\theta}} \left(v(k(\theta), \theta) - \frac{\partial v(k(\theta), \theta)}{\partial \theta} \cdot \frac{1 - \Phi(\theta)}{\phi(\theta)} \right) \phi(\theta) d\theta$$

and define $VS(\theta) = v(k(\theta), \theta) - \frac{\partial v(k(\theta), \theta)}{\partial \theta} \cdot \frac{1 - \Phi(\theta)}{\phi(\theta)}$ (note it's slightly different from how we defined VS in the linear case)

- And you can still find the optimal k by maximizing this virtual surplus (and it still has to be monotone)

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Optimal Mechanism: Lessons

- Incentives are costly.
 - If θ is an attractive type to imitate, have to distort θ 's allocation $k(\theta)$ compared to first-best (full-info benchmark).
 - (That's why $k(\bar{\theta})$ is not distorted.)
- Even though gains from trade always present, optimal to commit to not sell to low types to charge high types more.
- Distribution ϕ matters: if more high types then focus on them and sell with lower probability to the low types.
- It will most of the time be optimal to have some cutoff rule: $k(\theta) = \mathbb{I}\{\theta > \hat{\theta}\}$ for some $\hat{\theta}$.
 - Things become more interesting in multi-item case, see Manelli and Vincent [2007]

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Optimal Mechanism: Example

- Derive the optimal mechanism when $\theta \sim U[0, 1]$.
- $VS(\theta) = \theta - \frac{1-\Phi(\theta)}{\phi(\theta)} = 2\theta - 1$,
- hence optimal $k(\theta) = \mathbb{I}\{\theta \geq 0.5\}$. Payments?
- $U_b(\theta) = U_b(0) + \int_0^\theta k(\theta)d\theta = \max\{\theta - 0.5, 0\}$,
- so $t(\theta) = 0.5 \cdot \mathbb{I}\{\theta \geq 0.5\}$. Fixed price is optimal!
- Actually cannot do much better than fixed price in this simple trade model.
 - Maximizing $\int k(\theta)VS(\theta)d\theta$, which is linear in $k(\theta)$ for all θ . So we'll typically have either a cutoff rule, or constant rule – unless $VS(\theta)$ non-monotone.
 - Consequence of Euclidean payoffs. More interesting results with non-linear payoffs.

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Setting 3: many buyers, interval of types

- Designer/seller has one indivisible item for sale. Chooses allocation $k(\theta) \in \Delta(N)$ and payment profile $t(\theta) \in \mathbb{R}^N$ given report profile θ .
- Buyers $i \in \{1, \dots, N\}$ have valuations $\theta_i \sim \text{i.i.d.} \Phi[0, \bar{\theta}_i]$, private info.
 - Independence of θ_i is important, since we rely on revenue equivalence / ERP
- Buyer's preferences Euclidean: $u_b = \theta_i k_i - t_i$
- What is the optimal BIC mechanism that maximizes seller's expected profit?

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Optimal Auction

- We are effectively designing the optimal auction.
 - Selling the good to the highest bidder is efficient (assuming higher-value buyers bid more),
 - so all standard auction formats – first-/second-price, Dutch, English – are revenue-equivalent! (buyer with value zero gets zero)
- To get more profit often have to depart from efficiency, e.g. by
 - setting reservation price,
 - discriminating buyers (even if they are ex ante identical!).

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Optimal Auction

- From the perspective of the individual bidder, things are not much different from single-player model, just take expectations over θ_{-i} :

$$\bar{t}_i(\theta_i) \equiv \mathbb{E}_{\theta_{-i}} t_i(\theta_i, \theta_{-i})$$

$$\bar{k}_i(\theta_i) \equiv \mathbb{E}_{\theta_{-i}} k_i(\theta_i, \theta_{-i})$$

$$\bar{U}_i(\theta_i) \equiv \mathbb{E}_{\theta_{-i}} u_i(x(\theta_i, \theta_{-i}), \theta_i)$$

- Monotonicity: if $\theta'_i < \theta''_i$ then $\bar{k}_i(\theta'_i) \leq \bar{k}_i(\theta''_i)$.
- Envelope representation:

$$\bar{U}_i(\theta_i) = \bar{U}_i(0) + \int_0^{\theta_i} \bar{k}_i(s) ds.$$

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Optimal Auction: Seller

$$\begin{aligned} \mathbb{E} U_s &= \mathbb{E}_{\theta} \left[\sum_i t_i(\theta) \right] \\ &= \sum_i \mathbb{E}_{\theta} [\theta_i k_i(\theta) - U_i(\theta)] \\ &= \sum_i \mathbb{E}_{\theta_i} [\theta_i \bar{k}_i(\theta_i) - \bar{U}_i(\theta_i)] \\ &= \dots \\ &= \sum_i \left[\mathbb{E}_{\theta_i} \left[\bar{k}_i(\theta_i) \left(\theta_i - \frac{1 - \Phi_i(\theta_i)}{\phi_i(\theta_i)} \right) \right] - \bar{U}_i(0) \right] \\ &= \sum_i [\mathbb{E}_{\theta_i} \bar{k}_i(\theta_i) \textcolor{red}{VS}_i(\theta_i) - \bar{U}_i(0)] \end{aligned}$$

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Optimal Auction

- As before, set $\bar{U}_i(0) = 0$.

$$\begin{aligned}\mathbb{E}U_s &= \sum_i \mathbb{E}_{\theta_i} \bar{k}_i(\theta_i) VS_i(\theta_i) \\ &= \mathbb{E}_{\theta} \sum_i k_i(\theta) VS_i(\theta)\end{aligned}$$

- Pointwise maximization: for any θ , give the item to i with the highest $VS_i(\theta)$:

$$k_i(\theta) = \begin{cases} 1 & \text{if } i = \arg \max_j VS_j(\theta) \\ 0 & \text{otherwise} \end{cases}$$

(break ties as you wish)

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Optimal Auction: Conclusions

- Naive (pointwise) solution works only if the resulting allocations satisfy monotonicity.
 - If they don't: ???
 - Ironing is even more difficult because of joint constraint on allocations: $\sum_i k_i(\theta) \leq 1$.
- Allocations are inefficient:
 - Inefficient withholding when $\theta_i > 0$ but $VS_i < 0$ (and $i \in \arg \max_j VS_j$).
 - VS_i depend on respective distr-ns of θ_i 's – asymmetric players are treated asymmetrically.
- In **symmetric** case, the optimal auction can be implemented as one of standard formats (FPA, SPA, APA, Dutch, English) with reserve price.

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Optimal Contests

- Related topic: optimal contests.
 - N contestants exert effort, have private abilities.
 - Designer's goal: maximize total effort (e.g. maximize the amount of science that competing reserach teams produce).
 - How should designer choose size and number of prizes; winning rules etc?
- Will not cover in this class.

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References I

- A. Kleiner, B. Moldovanu, and P. Strack. Extreme Points and Majorization: Economic Applications. *Econometrica*, 89(4):1557–1593, 2021. ISSN 1468-0262. doi: 10.3982/ECTA18312.
- A. M. Manelli and D. R. Vincent. Multidimensional mechanism design: Revenue maximization and the multiple-good monopoly. *Journal of Economic Theory*, 137(1):153–185, November 2007. ISSN 0022-0531. doi: 10.1016/j.jet.2006.12.007.