

Mechanism Design

2: Efficient Mechanisms

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This slide deck:

- 1 Problem setup
- 2 VCG
- 3 Individual Rationality and Budget Balance
- 4 Payoff Equivalence
- 5 Payoff Equivalence in BIC
- 6 gVCG
- 7 AGV

Quasilinear Preferences

Assumption: Quasilinear setting/Transferable utility

- Instead of allowing all possible preferences, adopt a special structure.
- Instead of $x \in X$ describing everything related to outcome, split it into:
 - $k(\theta) \in K$, “real/material outcome” a.k.a. **allocation**
 - $t(\theta) \in \mathbb{R}^N$, **transfers/payments**
- Instead of arbitrary $u_i(x, \theta)$ focus on **quasilinear preferences**:

$$u_i(x, \theta_i) = v_i(k, \theta_i) - t_i$$

- S.c.f. is $f(\theta) = (k(\theta), t_1(\theta), \dots, t_N(\theta))$

Quasilinear Preferences

- Common interpretation: transfers=payments. This comes with a bunch of assumptions:
 - Monetary transfers always available,
 - individual utility is linear in money (risk-neutrality),
 - marginal social utility of money is constant across types and people and independent of allocation.
- All three are sometimes restrictive, the latter two especially.
- However, monetary payments are not necessary! Anything that i cares about that is not directly included in the allocation k can be used to adjust i 's utility as needed!
 - i 's time, i 's effort, promises to i , etc

Efficient Implementation

- A frequent question: “Dr.Professor, how can we as society implement the efficient outcome?”

- Reminder: efficient outcome $x^*(\theta) = (k^*(\theta), t^*(\theta))$ is

$$x^*(\theta) = \arg \max_x \sum_{i=0}^N u_i(x, \theta_i) = \arg \max_{(k,t)} \sum_{i=0}^N [v_i(k, \theta_i) - t_i]$$

- Transfers just reallocate utility across agents, so focus on efficient allocation $k^*(\theta)$:

$$k^*(\theta) = \arg \max_k \sum_{i=0}^N v_i(k, \theta_i)$$

- Note that we can include $i = 0$ into welfare calculations. This can capture designer's preferences or any social costs/benefits not captured by individual agents (e.g., cost of implementing a public project)

Efficient Implementation

- How do we do that?
 - We already know that it's enough to consider direct revelation mechanisms.
 - We have the desired allocation rule k^* , but we can design the transfers t – the problem is not just “check whether s.c.f. x^* is IC”, but “is there such t that k^* is IC?”
- What we as designers want:

$$\max \sum_{i=0}^N v_i(k, \theta_i)$$

- What agent i wants:

$$\max v_i(k, \theta_i) - t_i$$

- How to reconcile the two?

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VCG Mechanism: intro

- We now introduce the VCG mechanism that DSIC-implements the efficient allocation. We shall do it in a few steps.
- **NOTE:** while the broad idea behind the VCG mechanism is the same everywhere, the **exact definition** of the VCG mechanism **differs** in different sources (textbooks).

VCG Mechanism: Groves' Transfers

- More formally, the problem of agent i of type θ_i is:

$$\max_{\hat{\theta}_i} \left\{ v_i(k^*(\hat{\theta}_i, \theta_{-i}), \theta_i) - t_i(\hat{\theta}_i, \theta_{-i}) \right\}$$

- Try **Groves' transfers**:

$$t_i^G(\theta) \equiv - \left(\sum_{j \neq i} v_j(k^*(\theta_i, \theta_{-i}), \theta_j) \right) + h_i(\theta_{-i})$$

- Agent's problem is now

$$\max_{\hat{\theta}_i} \left\{ v_i(k^*(\hat{\theta}_i, \theta_{-i}), \theta_i) + \left(\sum_{j \neq i} v_j(k^*(\hat{\theta}_i, \theta_{-i}), \theta_j) \right) - h_i(\theta_{-i}) \right\}$$

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VCG Mechanism: Groves' Transfers

- Agent's problem is now

$$\max_{\hat{\theta}_i} \left\{ \sum_{j \in N} v_j(k^*(\hat{\theta}_i, \theta_{-i}), \theta_j) - h_i(\theta_{-i}) \right\}$$

- Every agent i chooses report $\hat{\theta}_i$ to maximize welfare!
 - Optimal to report true $\hat{\theta}_i$,
 - for any θ_{-i} .
- Crucial that $h_i(\theta_{-i})$ does not depend on i 's report.

VCG Mechanism: Example

Example (Moon Base)

- N citizens decide whether to build a Moon base which costs c
- citizen i has private valuation θ_i for the base and quasilinear utility (so if base built then $v_i = \theta_i$, otherwise $v_i = 0$)
- What are Groves' transfers? (Take $h_i(\theta_{-i}) \equiv 0$.)
- The incentives are there... but at what cost?

VCG Mechanism: Clarke Term

- A suggestion for $h_i(\theta_{-i})$ made by Clarke (“pivot mechanism”):

$$h_i(\theta_{-i}) = \sum_{j \neq i} v_j(k^{-i}(\theta_{-i}), \theta_j),$$

$$\text{where } k^{-i}(\theta_{-i}) \in \arg \max_k \sum_{j \neq i} v_j(k, \theta_j).$$

- **NOTE:** it is the **default allocation rules** $k^{-i}(\theta_{-i})$ that each textbook defines differently (or replaces with other things). My version is quite robust, but you can use other default rules if they make more sense in a given setting (so long as the rule for i is independent of θ_i)
- Resulting **VCG transfers:**

$$t_i^{\text{VCG}}(\theta) \equiv - \left(\sum_{j \neq i} v_j(k^*(\theta_i, \theta_{-i}), \theta_j) \right) + \sum_{j \neq i} v_j(k^{-i}(\theta_{-i}), \theta_j)$$

VCG Mechanism: Final Transfers

$$t_i^{VCG}(\theta) = - \left(\sum_{j \neq i} v_j(k^*(\theta_i, \theta_{-i}), \theta_j) \right) + \sum_{j \neq i} v_j(k^{-i}(\theta_{-i}), \theta_j)$$

- What's the big idea?
 - Agent i receives the externality his report imposes on others (mind the signs).
 - i 's transfer is non-zero only if his presence affects the decision k .
 - Note that i cannot misreport θ_i and get lower transfer without also changing k .
- What are VCG transfers in the Moon Base question?

VCG Mechanism: Example

Example (Auction)

- One indivisible item to be allocated among N bidders.
 - Bidder i 's valuation is θ_i (private info).
 - What is the VCG mechanism?
-
- VCG mechanism is the second-price auction (efficient and DSIC).
 - Also known as the Vickrey auction (the V in VCG).

VCG aftermath

- We have an easy recipe to implement the **efficient** outcome in **dominant** strategies.
- Any problems?

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Feature example: bilateral trade

Example (Bilateral Trade)

- One indivisible good.
- Two agents: buyer and seller.
- Private valuations $\theta_b, \theta_s \in [0, 1]$ resp.
- Find the VCG transfers (take no trade as efficient when $\theta_s = \theta_b$).

Feature example: bilateral trade

- If you did everything correctly, you'll get

$$t_b^{VCG}(\theta) = \theta_s \cdot \mathbb{I}\{\theta_s < \theta_b\}$$

$$t_s^{VCG}(\theta) = \theta_b \cdot \mathbb{I}\{\theta_s \geq \theta_b\}$$

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- The seller pays to keep the good and doesn't get anything from selling it. Good deal?

Individual rationality

- In many settings can't force players to participate in mechanism:

Definition (IR)

A mechanism Γ is:

- **interim individually rational** if $\mathbb{E}_{\theta_{-i}} [u_i(\theta_i, \theta_{-i})] \geq \underline{U}_i(\theta_i)$ for all θ_i ;
 - **ex post individually rational** if $u_i(\theta_i, \theta_{-i}) \geq \underline{U}_i(\theta_i)$ for all θ .
-
- $\underline{U}_i(\theta_i)$ is the outside option of type θ_i
 - (in bilateral trade: $\underline{U}_s(\theta_s) = \theta_s$)
 - expectation means that distribution of θ s now matters!
 - (whether a mechanism is DSIC does not depend on the distr-n; but whether it is IR does)

Detour – brief review

- **ex ante** = i knows nothing;
- **ex interim** = i knows θ_i ;
- **ex post** = i knows θ_i and θ_{-i} .
- We'll mostly work with interim IR;
- ex post IR is also sometimes used in the literature.

Budget balance

- VCG for bilateral trade example is not IR for seller (outside option = keep the good).

Budget balance

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- If we want mechanism to be IR, easy solution is to decrease $t_i(\theta)$ by a lot, for all θ .
- But that's expensive – want mechanism to be **budget balanced**:

Budget balance

- VCG for bilateral trade example is not IR for seller (outside option = keep the good).
- If we want mechanism to be IR, easy solution is to decrease $t_i(\theta)$ by a lot, for all θ .
- But that's expensive – want mechanism to be **budget balanced**:

Definition (BB)

- Mechanism Γ is **ex ante budget balanced** if $\mathbb{E}_\theta \left[\sum_{i=1}^N t_i(\theta) \right] \geq 0$;
 - Mechanism Γ is **ex post budget balanced** if $\sum_{i=1}^N t_i(\theta) \geq 0$ for all θ .
-
- Mechanism is **exactly BB** if the above hold with equalities.
 - If Γ is ex post BB then it is ex ante BB (prove).

IR vs BB

- Fundamental tension between IR and BB.
- We want to ask the following question (within our [bilateral trade](#) example, in particular):

Does there exist a mechanism that is: [efficient](#), [DSIC](#), [IR](#), [BB](#)?

- Fundamental tension between IR and BB.
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Does there exist a mechanism that is: [efficient](#), [DSIC](#), [IR](#), [BB](#)?

- We know VCG was not IR, but that's just one mechanism. Can we say whether any other mechanisms satisfy all requirements?
 - Not in most general case*, but all examples (trade, auction, pub.project) fit a much narrower model where we can.
 - *though see Prop 23.C.5 in MWG

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The Euclidean model

Assumption: Euclidean setting

Make the following assumptions on top of quasilinearity:

- $\theta_i \in \Theta_i = [\underline{\theta}_i, \bar{\theta}_i] \subseteq \mathbb{R}$, full support;
- $k \in K \subseteq \mathbb{R}^N$, K compact, convex set;
- $u_i(x, \theta_i) = \theta_i k_i - t_i$.

- I'll call the above **the Euclidean model** (not standard name).
- We'll derive **Payoff-equivalence** as a necessary condition for Γ to be **DSIC** in **Euclidean** model. It's a cool property on its own and will help answer the question about BB/IR mechanisms later.
- Given Γ , denote $U_i(\theta_i, \theta_{-i}) \equiv u_i(x(\theta_i, \theta_{-i}), \theta_i)$.

Monotonicity

- Assume Γ is a **direct** mechanism (or consider its direct equivalent).
- Play a bit with i 's IC (truthtelling constraint): for any $i, \theta_i, \hat{\theta}_i, \theta_{-i}$,

$$U_i(\theta_i, \theta_{-i}) \geq u_i(x(\hat{\theta}_i, \theta_{-i}), \theta_i)$$

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- In the end:

$$U_i(\theta_i, \theta_{-i}) \geq U_i(\hat{\theta}_i, \theta_{-i}) + (\theta_i - \hat{\theta}_i) k_i(\hat{\theta}_i, \theta_{-i}).$$

- Similarly, type $\hat{\theta}_i$ should not want to report θ_i :

$$U_i(\hat{\theta}_i, \theta_{-i}) \geq U_i(\theta_i, \theta_{-i}) + (\hat{\theta}_i - \theta_i) k_i(\theta_i, \theta_{-i}).$$

Monotonicity

- Combining the two for $\theta_i > \hat{\theta}_i$, we get

$$k_i(\theta_i, \theta_{-i}) \geq \frac{U_i(\theta_i, \theta_{-i}) - U_i(\hat{\theta}_i, \theta_{-i})}{\theta_i - \hat{\theta}_i} \geq k_i(\hat{\theta}_i, \theta_{-i}), \quad (1)$$

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- meaning $k_i(\theta_i, \theta_{-i}) \geq k_i(\hat{\theta}_i, \theta_{-i})$ – allocation rule must be **monotone**.
- DSIC: “Those who value things more should get more things.”
- **Monotonicity** is **necessary** for f to be **DSIC** in **Euclidean** settings.
- From monotonicity we can build up to **payoff equivalence**, the second cool result in mechanism design (after revelation principle, not monotonicity).

Payoff Equivalence

- $k_i(\theta_i, \theta_{-i})$ is monotone in θ_i , hence continuous a.e.: $\lim_{\hat{\theta}_i \rightarrow \theta_i} k_i(\hat{\theta}_i, \theta_{-i}) = k_i(\theta_i, \theta_{-i})$.

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$$\frac{\partial U_i(\theta_i, \theta_{-i})}{\partial \theta_i} = \lim_{\hat{\theta}_i \rightarrow \theta_i} \frac{U_i(\theta_i, \theta_{-i}) - U_i(\hat{\theta}_i, \theta_{-i})}{\theta_i - \hat{\theta}_i} = k_i(\theta_i, \theta_{-i}).$$

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- So if $k(\theta)$ is integrable in θ_i (e.g. if it's bounded) then for all θ_i

$$U_i(\theta_i, \theta_{-i}) = U_i(\underline{\theta}_i, \theta_{-i}) + \int_{\underline{\theta}_i}^{\theta_i} k_i(s, \theta_{-i}) ds$$

(Note that the lower limit does not need to be $\underline{\theta}_i$ – it can be any other type.)

- This is the **envelope representation of payoffs** a.k.a. Mirrlees condition. From it we can immediately get revenue equivalence.

Payoff Equivalence

Theorem (Payoff Equivalence for DSIC Euclidean mechanisms)

*In the Euclidean setting, for any two DSIC DRMs with $x = (k, t)$ and $x' = (k', t')$ respectively:
if $k(\theta) = k'(\theta)$ for all θ then $t_i(\theta) = t'_i(\theta) + c_i(\theta_{-i})$ for all θ for some $c_i(\theta_{-i})$.*

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Proof. Given the envelope representation, invoke the definition of U_i :

$$v_i(k^*(\theta_i, \theta_{-i}), \theta_i) - t_i(\theta_i, \theta_{-i}) = U_i(\underline{\theta}_i, \theta_{-i}) + \int_{\underline{\theta}_i}^{\theta_i} k_i(s, \theta_{-i}) ds.$$

The above holds in any DSIC DRM. Take the two mechanisms in the statement, fix some θ, i , express $t_i(\theta)$ and $t'_i(\theta)$ from the above, and you will get that

$$t_i(\theta) = t'_i(\theta) - U_i(\underline{\theta}_i, \theta_{-i}) + U'_i(\underline{\theta}_i, \theta_{-i}),$$

where U_i and U'_i denote the eqm utilities in the two mechanisms. The last two terms only depend on i and θ_{-i} (but not θ_i), hence denoting them as $c_i(\theta_{-i})$ concludes the proof. \square

Payoff Equivalence: intuition

- Given allocation k (doesn't have to be efficient), utility of one type (usually “lowest” type) pins down utils of all types of player i given fixed θ_{-i} .

Payoff Equivalence: intuition

- Given allocation k (doesn't have to be efficient), utility of one type (usually “lowest” type) pins down utils of all types of player i given fixed θ_{-i} .
- Equivalently, have only one degree of freedom for i 's transfers given θ_{-i} .
- Reminds you of anything?

Payoff Equivalence of Efficient Mechanisms

- Recall Groves' transfers: efficient k^* can be impl-d in DS by

$$\begin{aligned} t_i(\theta) &= - \left(\sum_{j \neq i} v_j(k^*(\theta_i, \theta_{-i}), \theta_j) \right) + h_i(\theta_{-i}) \\ &= - \left(\sum_{j \neq i} \theta_j k_j^*(\theta_i, \theta_{-i}) \right) + h_i(\theta_{-i}) \end{aligned}$$

for some $h_i(\theta_{-i})$.

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for some $h_i(\theta_{-i})$.

- Payoff equivalence implies that **efficient** k^* in a **Euclidean** model can **ONLY** be DS-implemented by some **Groves'** mechanism.

Payoff Equivalence: Beyond Euclidean

Both monotonicity and payoff equivalence hold beyond the Euclidean setting:

- various forms of **monotonicity** are necessary and sufficient for DSIC in quasilinear setting;
- **payoff equivalence** of DSIC mechanisms is generalizable beyond Euclidean (but you cannot get to general quasilinear setting);
- see Börgers for details:
 - ch.5: single-player problems,
 - ch.7: DSIC results building on ch.5.

Payoff Equivalence (DSIC): Conclusion

- So what does payoff equivalence tell us?
- Efficient allocation k^* can **only** be implemented using Groves' transfers...
 - ...but $h_i(\theta_{-i})$ still provides a lot of flexibility!
 - So it's hard to know whether VCG is the best mechanism or there are others.
- So let us *weaken* our implementation concept to obtain a **stronger** version of payoff equivalence.

Back to bilateral trade

- Remember how this detour started?

Example (Bilateral Trade)

- One indivisible good.
- Two agents: buyer and seller.
- Private valuations $\theta_b, \theta_s \in [0, 1]$ resp.
- Is there an efficient, DSIC, ex post IR, ex post BB mechanism?

Can we answer this question now?

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Payoff Equivalence in BIC

We now show payoff equivalence for BIC mechanisms in the Euclidean setting.

New **assumption**: types are independent across players: $\theta_i \perp \theta_{-i}$ for all i .

Theorem (Payoff Equivalence for BIC mechanisms)

For any two BIC DRMs with $x = (k, t)$ and $x' = (k', t')$ resp.:

if $\mathbb{E}_{\theta_{-i}} k_i(\theta_i, \theta_{-i}) = \mathbb{E}_{\theta_{-i}} k'_i(\theta_i, \theta_{-i})$ for all i, θ_i ,

then $\mathbb{E}_{\theta_{-i}} t_i(\theta_i, \theta_{-i}) = \mathbb{E}_{\theta_{-i}} t'_i(\theta_i, \theta_{-i}) + h_i$ for all i, θ_i for some h_i .

- As before, implies that for given k (any, not just the efficient) we only have one degree of freedom for $t_i(\theta)$,
 - now “just one” instead of “just one given θ_{-i} ”.

Payoff Equivalence in BIC. Proof

■ Let

$$\begin{aligned}\tilde{U}_i(\hat{\theta}_i, \theta_i) &\equiv \mathbb{E}_{\theta_{-i}} \left[u_i \left(x(\hat{\theta}_i, \theta_{-i}), \theta_i \right) \mid \theta_i \right] \\ &= \mathbb{E}_{\theta_{-i}} \left[\theta_i k_i(\hat{\theta}_i, \theta_{-i}) - t_i \left(\hat{\theta}_i, \theta_{-i} \right) \mid \theta_i \right].\end{aligned}$$

(do not confuse with U_i in Euclidean model for DS.)

Payoff Equivalence in BIC. Proof

- Let

$$\begin{aligned}\tilde{U}_i(\hat{\theta}_i, \theta_i) &\equiv \mathbb{E}_{\theta_{-i}} \left[u_i \left(x(\hat{\theta}_i, \theta_{-i}), \theta_i \right) \mid \theta_i \right] \\ &= \mathbb{E}_{\theta_{-i}} \left[\theta_i k_i(\hat{\theta}_i, \theta_{-i}) - t_i \left(\hat{\theta}_i, \theta_{-i} \right) \mid \theta_i \right].\end{aligned}$$

(do not confuse with U_i in Euclidean model for DS.)

- Take full derivative w.r.t θ_i at $\hat{\theta}_i = \theta_i$:

$$\begin{aligned}\frac{d}{d\theta_i} \tilde{U}_i(\theta_i, \theta_i) &= \frac{\partial}{\partial \hat{\theta}_i} \tilde{U}_i(\hat{\theta}_i, \theta_i) \Big|_{\hat{\theta}_i=\theta_i} + \frac{\partial}{\partial \theta_i} \tilde{U}_i(\hat{\theta}_i, \theta_i) \Big|_{\hat{\theta}_i=\theta_i} \\ &= 0 + \mathbb{E}_{\theta_{-i}} [k_i(\theta_i, \theta_{-i}) \mid \theta_i]\end{aligned}$$

The first term is zero because truthful report $\hat{\theta}_i = \theta_i$ maximizes $\tilde{U}_i(\hat{\theta}_i, \theta_i)$.

Payoff Equivalence in BIC. Proof

- Then by the Fundamental Theorem of Calculus (with $\bar{U}_i(\theta_i) \equiv \tilde{U}_i(\theta_i, \theta_i)$)

$$\begin{aligned}\bar{U}_i(\theta_i) &= \bar{U}_i(\underline{\theta}_i) + \int_{\underline{\theta}_i}^{\theta_i} \frac{d\bar{U}_i}{d\theta_i}(s, s) ds \\ &= \bar{U}_i(\underline{\theta}_i) + \int_{\underline{\theta}_i}^{\theta_i} \mathbb{E}_{\theta_{-i}} [k_i(s, \theta_{-i}) | \theta_i] ds,\end{aligned}$$

meaning that k and $\bar{U}_i(\underline{\theta}_i)$ pin down utilities $\bar{U}_i(\theta_i)$ for all θ_i . \square

- Remark: here we used a different argument to get $\frac{d\bar{U}_i(\theta_i, \theta_i)}{d\theta_i} = \mathbb{E}_{\theta_{-i}} [k(\theta_i, \theta_{-i}) | \theta_i]$ compared to DSIC proof. Either argument can be used in both proofs.

Payoff Equivalence in BIC. Generalization

The proof is nice and illustrative in Euclidean setting.

Krishna and Maenner [2001] present a more general result for the following setting:

- quasilinear setting;
- independent types;
- $\Theta_i \subseteq \mathbb{R}^{K_i}$ is a convex set for every i
- $v_i(k, \theta_i)$ is convex in θ_i for all i .

Ex ante and ex post revenue in BIC

One cool thing about BIC mechanisms is that ex post BB is free if you have ex ante BB:

Theorem

*In a **quasilinear** setting, for every direct mechanism $\Gamma = (\Theta, (k, t))$ that is BIC and **ex ante BB**, there exists a direct mechanism $\Gamma' = (\Theta, (k', t'))$ which is:*

- *BIC,*
- ***ex post BB**,*
- *equivalent to Γ in the sense of: $k'(\theta) = k(\theta)$ for all θ and $\mathbb{E}_{\theta_{-i}} t'_i(\theta_i, \theta_{-i}) = \mathbb{E}_{\theta_{-i}} t_i(\theta_i, \theta_{-i})$ for all i and θ_i .*

For proof, see Prop 6.3 & Prop 3.6 in B"orger's.

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Generalized VCG

- As it turns out, VCG can be (interim) IR with a slight modification... And it will be (ex ante) revenue-maximizing among all such mechanisms in that case.
- Enter **generalized VCG** [Krishna and Perry, 2000].

Generalized VCG

- As it turns out, VCG can be (interim) IR with a slight modification... And it will be (ex ante) revenue-maximizing among all such mechanisms in that case.
- Enter **generalized VCG** [Krishna and Perry, 2000].
- Define **least charitable type** $\tilde{\theta}_i$ as

$$\tilde{\theta}_i \in \arg \min_{\theta_i \in \Theta_i} \mathbb{E}_{\theta_{-i}} \left[\sum_{j=0}^N v_j(k^*(\theta_i, \theta_{-i}), \theta_j) - \underline{U}_i(\theta_i) \right]$$

(expectation taken w.r.t the common prior $\phi \in \Delta(\Theta)$)

Generalized VCG

GVCG mechanism is a DRM with the efficient allocation $k^*(\theta)$ and payments

$$t_i^{GVCG}(\theta) \equiv \sum_{j \neq i} v_j(k^*(\tilde{\theta}_i, \theta_{-i}), \theta_j) + v_i(k^*(\tilde{\theta}_i, \theta_{-i}), \tilde{\theta}_i) - \\ - \sum_{j \neq i} v_j(k^*(\theta_i, \theta_{-i}), \theta_j) - \underline{U}_i(\tilde{\theta}_i)$$

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Has the usual Groves' term (the third one); the other three guarantee IR.

- The first term is similar to Clarke's term, but with $k^*(\tilde{\theta}_i, \theta_{-i})$ instead of $k^{-i}(\theta_{-i})$
- 2^{nd} and 4^{th} is the net utility that LCT $\tilde{\theta}_i$ gets from participating in the mechanism – need to also pay it to all other types

Theorem (gVCG, part 1)

In a *quasilinear* model, gVCG is:

- *efficient (by construction),*
- *DSIC,*
- *interim IR.*
-

Prove DSIC on your own (analogous to VCG).

Generalized VCG. Proof: IIR

Interim expected utility for θ_i is

$$\mathbb{E}_{\theta_{-i}} \left[\sum_{j=0}^N v_j(k^*(\theta), \theta_j) - \sum_{j=0}^N v_j(k^*(\tilde{\theta}_i, \theta_{-i}), \theta_j) \middle|_{\theta_i = \tilde{\theta}_i} \middle| \theta_i \right] + \underline{U}_i(\tilde{\theta}_i)$$

Generalized VCG. Proof: IIR

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The inequality above is the IIR constraint, and it holds since

$$\tilde{\theta}_i \in \arg \min_{\theta_i \in \Theta_i} \mathbb{E}_{\theta_{-i}} \left[\sum_{j=0}^N v_j(k^*(\theta_i, \theta_{-i}), \theta_j) - \underline{U}_i(\theta_i) \middle| \theta_i \right]$$

Theorem (gVCG, part 2)

In a *Euclidean* model with independent players' types, gVCG is:

- efficient (by construction),
- DSIC,
- interim IR;
- maximizes expected revenue among all mechanisms that are BIC, IIR, and implement the efficient k^* .

If gCVG is not ex ante budget balanced, there does not exist a
{EFF + BIC + IIR + ex ante BB} mechanism (so no ex post BB either).

Generalized VCG. Proof: revenue maximizing in Euclidean

- Given revenue equivalence, just need to show we cannot decrease h_i for any player w/o violating IIR.

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- Decreasing h_i only possible if IR slack for *all* types of i .

Generalized VCG. Proof: revenue maximizing in Euclidean

- Given revenue equivalence, just need to show we cannot decrease h_i for any player w/o violating IIR.
- Decreasing h_i only possible if IR slack for *all* types of i .
- But IR binds for $\tilde{\theta}_i$: $\bar{U}_i(\tilde{\theta}_i) = \underline{U}_i(\tilde{\theta}_i)$ (verify).



Application: Bilateral Trade

Example (Bilateral Trade (revisited))

- One indivisible good.
 - Two agents: buyer and seller.
 - Private valuations $\theta_b, \theta_s \sim \text{i.i.d. } U[0, 1]$ resp.
 - Is there an ~~efficient, DSIC, ex post IR, ex post BB~~ efficient, BIC, interim IR, ex ante BB mechanism?
-
- No, because gVCG is not BB. (This is the [Myerson-Satterthwaite Theorem](#))

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AGV mechanism

- One last mechanism before we go – in case you care about BB, but not IR.
- Let

$$\tilde{t}_i(\theta_i) \equiv \mathbb{E}_{\theta_{-i}} \left[\sum_{j \neq i} v_j(k^*(\theta_i, \theta_{-i}), \theta_j) | \theta_i \right]$$

be the “expected externality” imposed by i on everyone else.

- AGV transfers are given by

$$t_i^{AGV}(\theta) \equiv \frac{1}{N-1} \sum_{j \neq i} \tilde{t}_j(\theta_j) - \tilde{t}_i(\theta_i).$$

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$$t_i^{AGV}(\theta) \equiv \frac{1}{N-1} \sum_{j \neq i} \tilde{t}_j(\theta_j) - \tilde{t}_i(\theta_i).$$

- The second term is the averaged version of Groves' transfer,
- the first term is $h_i(\theta_{-i})$ which balances the budget.

Theorem (AGV)

In a *quasilinear* model, AGV is:

- efficient (by construction),
- exactly ex post BB,
- BIC.

Not necessarily IR. :(

AGV mechanism. Proof: budget balance.

- Observe that

$$\sum_i t_i^{\text{AGV}}(\theta) = \sum_i \left[\frac{1}{N-1} \sum_{j \neq i} \tilde{t}_j(\theta_j) - \tilde{t}_i(\theta_i) \right].$$

- For any j , RHS has:

- $N - 1$ terms of the form $\frac{1}{N-1} \tilde{t}_j(\theta_j)$, and

- 1 term of the form $-\tilde{t}_j(\theta_j)$.

- These cancel out and exhaust all terms in the sum. Therefore, $\sum_i t_i^{\text{AGV}}(\theta) = 0$ for all $\theta =$ ex post exact budget balance.

- **Note:** if the mechanism needs to raise some fixed sum *for any* θ , it can be treated as \tilde{t}_0 .

- If the mechanism needs to raise some sum that is *dependent on* θ (e.g. fund a public project iff it is built), AGV **cannot** handle that.

AGV mechanism. Proof: BIC.

- If i reports $\hat{\theta}_i$ then receives utility

$$\mathbb{E}_{\theta_{-i}} \left[v_i(k^*(\hat{\theta}_i, \theta_{-i}), \theta_i) + \sum_{j \neq i} v_j(k^*(\hat{\theta}_i, \theta_{-i}), \theta_j) | \theta_i \right] - \frac{1}{N-1} \sum_{j \neq i} \tilde{t}_j(\theta_j)$$

- Last term indep of $\hat{\theta}_i$;

bracket max-d by $\hat{\theta}_i = \theta_i$ for every θ_{-i} (since k^* efficient),

so max-d by $\hat{\theta}_i = \theta_i$ in expectation as well.

- Reporting truth is a best response to $-i$ reporting truthfully

\Rightarrow truthful reporting is a BNE of the mechanism.



- We have two mechanisms that implement the efficient k^* in quasilinear model:
 - AGV: BIC + BB,
 - gVCG: DSIC + IIR.
- In the Euclidean model, gVCG is also [ex ante-]revenue-maximizing among BIC+IIR mechanisms.
- Revenue equivalence is powerful, but needs more structure than just quasilinear model.

References I

- V. Krishna and E. Maenner. Convex potentials with an application to mechanism design. *Econometrica*, 69(4):1113–1119, 2001.
- V. Krishna and M. Perry. Efficient Mechanism Design. *SSRN Electronic Journal*, 2000. ISSN 1556-5068. doi: 10.2139/ssrn.64934.