Mechanism Design

3: Optimal (revenue-maximizing) mechanisms

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Done so far

- Introduced basic notions and criteria:
  - s.c.f., mechanism, implementation and implementability (DSIC, BIC),
  - efficiency, individual rationality, budget balance.

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  - revelation principle (pretty universal),
  - payoff/revenue equivalence (Euclidean model, slightly generalizable),
  - necessary conditions for implementability (weak preference reversal, monotonicity)

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- Covered some fundamental results in Mechanism Design:
  - revelation principle (pretty universal),
  - payoff/revenue equivalence (Euclidean model, slightly generalizable),
  - necessary conditions for implementability (weak preference reversal, monotonicity)
- Learned to implement the efficient s.c.f.:
  - DSIC: VCG;
  - BIC: AGV, gVCG.

Today

- Finished with implementing efficient s.c.f.s
- Today will look at revenue maximization.
  - Revenue-maximizing mechanisms called "optimal" in the literature (meaning optimal for the designer), after Myerson's "optimal mechanism".
- gVCG was optimal in the class of efficient mechanisms. Now we remove the restriction on allocations.

This slide deck:

1. Two types (Monopolistic Screening)
2. Interval of types (Optimal Mechanism)
3. Many buyers (Optimal Auction)

Setting 1: one buyer, discrete type

- Starting simple (Monopolistic Screening / Second-Degree Price Discrimination).
- Seller-designer can set quantities k and prices t for product, has production costs $c(k) = k^2 (= -v_0(k))$.
  - As usual, designer has no private information. "Informed principal" is a difficult problem.
- There is one buyer with valuation $\theta \in \{L, H\}$, private info. Prior probabilities are $\phi(H) = \phi$, $\phi(L) = 1 - \phi$.
- Buyer's preferences Euclidean: $u_b(x, \theta) = \theta k - t$
  - Is this a Euclidean model?

Monopolistic Screening

- As usual, look at DRM $\Gamma = (\Theta, (k, t))$. Notation-wise, let $k_\theta \equiv k(\theta)$ and $t_\theta \equiv t(\theta)$.
- Seller's problem (contrary to before, we can now choose k in addition to t.)

$$\max_{(k,t)} \left\{\phi(t_H - k_H^2) + (1-\phi)(t_L - k_L^2)\right\}$$

$$\begin{aligned} \text{s.t. } (IC_H):&\quad \theta_H k_H - t_H \geq \theta_H k_L - t_L \\ (IC_L):&\quad \theta_L k_H - t_H \leq \theta_L k_L - t_L \\ (IR_H):&\quad \theta_H k_H - t_H \geq 0 \\ (IR_L):&\quad \theta_L k_L - t_L \geq 0 \end{aligned}$$

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5. solve for optimal (k, t).

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Lessons:

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1. offering a menu may be optimal to extract value from buyers;
2. explains weird non-linear prices you can often encounter;
3. quantity is distorted downward for low type
4. high type gets information rent (pays below valuation);
5. IR must bind for at least some type.

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2 Interval of types (Optimal Mechanism)

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Setting 2: one buyer, interval of types

- Designer/seller has one indivisible item for sale. Chooses menu including probability of sale $k(\theta) \in [0,1]$ and payment $t(\theta)$ given report θ, no costs for simplicity.
  - Nothing changes from when k was quantity, since everyone is risk-neutral.
- Buyer has valuation $\theta \sim \Phi[0, \bar{\theta}]$, private info.
- Buyer's preferences Euclidean: $u_b = \theta k - t$.
- Buyer's outside option yields utility zero: $\underline{U}_b(\theta) = 0$.

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&= \int_0^{\bar{\theta}} k(\theta)\left(\theta - \frac{1-\Phi(\theta)}{\phi(\theta)}\right)\phi(\theta)d\theta - U_b(0)
\end{aligned}
$$

Optimal Mechanism: integration by parts

Integration by parts under the microscope:

$$\begin{aligned}
\int_0^{\bar\theta} \left(\int_0^{\theta} k(s)ds \right) \phi(\theta)d\theta &= \left[\Phi(\theta) \int_0^{\theta} k(s)ds \right]\Bigg|_{\theta=0}^{\bar\theta} - \int_0^{\bar\theta} F(\theta)k(\theta)d\theta \\
&= \int_0^{\bar\theta} k(\theta)d\theta - \int_0^{\bar\theta} \Phi(\theta)k(\theta)d\theta \\
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Optimal Mechanism: pinning $U_b(0)$

$$\mathbb{E}U_s = \mathbb{E}_\theta[t(\theta)] = \int_0^{\bar{\theta}} k(\theta)\left(\theta - \frac{1-\Phi(\theta)}{\phi(\theta)}\right)\phi(\theta)d\theta - U_b(0)$$

- To choose: allocation rule $k(\theta)$ and $U_b(0)$ (pins transfers).

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- What do with $U_b(0)$?
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  - Gotta be $U_b(0) \geq 0$ to satisfy IR for $\theta = 0$. Other types?

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  - Recall $U_b(\theta) = U_b(0) + \int_0^\theta k(s)ds$ and $k(\theta) \geq 0$, so $U_b(\theta) \geq U_b(0)$ for all θ,
  - hence $U_b(0) = \underline{U}_b(\theta) = 0$ is optimal (max revenue, all IR hold, IR binds for $\theta = 0$).

Optimal Mechanism: optimal k

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  - Define virtual surplus $VS(\theta) := \theta - \frac{1-\Phi(\theta)}{\phi(\theta)}$.

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$$\mathbb{E}U_s = \int_0^{\bar{\theta}} k(\theta) \left(\theta - \frac{1 - \Phi(\theta)}{\phi(\theta)} \right) \phi(\theta) d\theta$$

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  - Remember: $k(\theta)$ is only implementable if it is monotone! Sufficient condition: $VS(\theta)$ increasing in θ.

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  - Remember: $k(\theta)$ is only implementable if it is monotone! Sufficient condition: $VS(\theta)$ increasing in θ.
  - In the end, if $VS(\theta)$ is increasing in θ, the optimal mechanism is given by $k(\theta)$ as above and $t(\theta)$ that can be computed from ERP.

Optimal Mechanism: Virtual Surplus

- What is virtual surplus?
  - It reflects information rents we have to pay to high types to incentivize them to reveal type honestly.
- Sufficient for increasing $VS(\theta)$ is increasing hazard rate $\frac{\phi(\theta)}{1-\Phi(\theta)}$.
  The assumption we usually live with; need suitable distribution $\Phi(\theta)$.
- What do if "unlucky" and $\Phi(\theta)$ is such that VS is sometimes decreasing?
  - "Ironing": find monotone $k(\theta)$ that is "closest" to the unconstrained optimum.
  - E.g. if VS is globally decreasing then some constant k is optimal.
  - There is a kind of a general approach to this, but it's difficult, see Kleiner, Moldovanu, and Strack [2021].

Optimal Mechanism: non-linear preferences

- Note that linear preferences $v(k,\theta) = \theta k$ are not necessary for any of this.
- With general v you will not get a nice decomposition $k \cdot VS$ in the integral.
- But you can still obtain something like

$$\int_0^{\bar{\theta}} \left(v(k(\theta),\theta) - \frac{\partial v(k(\theta),\theta)}{\partial \theta} \cdot \frac{1-\Phi(\theta)}{\phi(\theta)} \right) \phi(\theta) d\theta$$

  and define $VS(\theta) = v(k(\theta),\theta) - \frac{\partial v(k(\theta),\theta)}{\partial \theta} \cdot \frac{1-\Phi(\theta)}{\phi(\theta)}$ (note it's slightly different from how we defined VS in the linear case)
- And you can still find the optimal k by maximizing this virtual surplus (and it still has to be monotone)

Optimal Mechanism: Lessons

- Incentives are costly.
  - If θ is an attractive type to imitate, have to distort θ's allocation $k(\theta)$ compared to first-best (full-info benchmark).
  - (That's why $k(\bar{\theta})$ is not distorted.)
- Even though gains from trade always present, optimal to commit to not sell to low types to charge high types more.
- Distribution ϕ matters: if more high types then focus on them and sell with lower probability to the low types.
- It will most of the time be optimal to have some cutoff rule: $k(\theta) = \mathbb{I}\{\theta > \hat{\theta}\}$ for some $\hat{\theta}$.
  - Things become more interesting in multi-item case, see Manelli and Vincent [2007]

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  - Maximizing $\int k(\theta) VS(\theta) d\theta$, which is linear in $k(\theta)$ for all θ. So we'll typically have either a cutoff rule, or constant rule – unless $VS(\theta)$ non-monotone.
  - Consequence of Euclidean payoffs. More interesting results with non-linear payoffs.

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2. Interval of types (Optimal Mechanism)
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Setting 3: many buyers, interval of types

- Designer/seller has one indivisible item for sale. Chooses allocation $k(\theta) \in \Delta(N)$ and payment profile $t(\theta) \in \mathbb{R}^N$ given report profile θ.
- Buyers $i \in \{1, ..., N\}$ have valuations $\theta_i \sim$ i.i.d.$\Phi[0, \bar{\theta}_i]$, private info.
  - Independence of θ_i is important, since we rely on revenue equivalence / ERP
- Buyer's preferences Euclidean: $u_b = \theta_i k_i - t_i$
- What is the optimal BIC mechanism that maximizes seller's expected profit?

Optimal Auction

- We are effectively designing the optimal auction.
  - Selling the good to the highest bidder is efficient (assuming higher-value buyers bid more),
  - so all standard auction formats – first-/second-price, Dutch, English – are revenue-equivalent! (buyer with value zero gets zero)
- To get more profit often have to depart from efficiency, e.g. by
  - setting reservation price,
  - discriminating buyers (even if they are ex ante identical!).

Optimal Auction

- From the perspective of the individual bidder, things are not much different from single-player model, just take expectations over θ_{-i}:

$$\bar{t}_i(\theta_i) \equiv \mathbb{E}_{\theta_{-i}} t_i(\theta_i, \theta_{-i})$$
$$\bar{k}_i(\theta_i) \equiv \mathbb{E}_{\theta_{-i}} k_i(\theta_i, \theta_{-i})$$
$$\bar{U}_i(\theta_i) \equiv \mathbb{E}_{\theta_{-i}} u_i(x(\theta_i, \theta_{-i}), \theta_i)$$

- Monotonicity: if $\theta_i' < \theta_i''$ then $\bar{k}_i(\theta_i') \leq \bar{k}_i(\theta_i'')$.
- Envelope representation:

$$\bar{U}_i(\theta_i) = \bar{U}_i(0) + \int_0^{\theta_i} \bar{k}_i(s) ds.$$

Optimal Auction: Seller

$$
\begin{aligned}
\mathbb{E}U_s &= \mathbb{E}_\theta \left[\sum_i t_i(\theta) \right] \\
&= \sum_i \mathbb{E}_\theta \left[\theta_i k_i(\theta) - U_i(\theta) \right] \\
&= \sum_i \mathbb{E}_{\theta_i} \left[\theta_i \bar{k}_i(\theta_i) - \bar{U}_i(\theta_i) \right] \\
&= \ldots \\
&= \sum_i \left[\mathbb{E}_{\theta_i} \left[\bar{k}_i(\theta_i) \left(\theta_i - \frac{1 - \Phi_i(\theta_i)}{\phi_i(\theta_i)} \right) \right] - \bar{U}_i(0) \right] \\
&= \sum_i \left[\mathbb{E}_{\theta_i} \bar{k}_i(\theta_i) VS_i(\theta_i) - \bar{U}_i(0) \right]
\end{aligned}
$$

Optimal Auction

- As before, set $\bar{U}_i(0) = 0$.

$$\begin{aligned}\mathbb{E}U_s &= \sum_i \mathbb{E}_{\theta_i} \bar{k}_i(\theta_i) VS_i(\theta_i) \\ &= \mathbb{E}_\theta \sum_i k_i(\theta) VS_i(\theta)\end{aligned}$$

- Pointwise maximization: for any θ, give the item to i with the highest $VS_i(\theta)$:

$$k_i(\theta) = \begin{cases} 1 & \text{if } i = \arg\max_j VS_j(\theta) \\ 0 & \text{otherwise} \end{cases}$$

(break ties as you wish)

Optimal Auction: Conclusions

- Naive (pointwise) solution works only if the resulting allocations satisfy monotonicity.
  - If they don't: ???
  - Ironing is even more difficult because of joint constraint on allocations: $\sum_i k_i(\theta) \leq 1$.
- Allocations are inefficient:
  - Inefficient withholding when $\theta_i > 0$ but $VS_i < 0$ (and $i \in \arg\max_j VS_j$).
  - VS_i depend on respective distr-ns of θ_i's – asymmetric players are treated asymmetrically.
- In **symmetric** case, the optimal auction can be implemented as one of standard formats (FPA, SPA, APA, Dutch, English) with reserve price.

Optimal Contests

- Related topic: optimal contests.
  - N contestants exert effort, have private abilities.
  - Designer's goal: maximize total effort (e.g. maximize the amount of science that competing reserach teams produce).
  - How should designer choose size and number of prizes; winning rules etc?
- Will not cover in this class.

References I

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