Exercises for Lecture 3 (M2): Efficient Mechanisms

Problem 1: Combinatorial Auction

There are two items, $l = A, B$ (e.g., two adjacent plots of land), being auctioned off among two bidders, $i = 1, 2$. Each bidder's valuations for the two items and the bundle consisting of both items are $(v_{i,A}, v_{i,B}, v_{i,AB}) \sim F_i$, distributed independently across bidders; the auctioneer's value for the items is zero. Their value of not getting either item is zero, and their utility functions are quasilinear in payments.

The auctioneer runs a VCG auction to allocate the items. In this auction, each bidder reports a bid for every combination of items. The resulting allocation is determined so as to maximize the bidders' welfare according to their reported valuations, and the payments are determined using the VCG payment rule.

The bidders' *realized* valuations are described by Table 1 (note that these valuations are still bidders' private information).

	$\{A\}$	$\{B\}$	$\{A, B\}$
Bidder 1	20	11	33
Bidder 2	14	14	29

Table 1: Realized valuations for the combinatorial auction problem

Calculate the resulting allocation and payments.

Problem 2: Efficient public good provision

This is a more standard problem on VCG mechanism. A version of this problem with numbers has been done in lecture (Moon base example), now do it with letters.

There is a society of N people. They must collectively decide whether to implement a public project (e.g., build a bridge, or pass a tax reform, or order a pizza). Let $k \in \{0, 1\}$ denote the outcome of this decision: $k = 1$ if project is implemented, $k = 0$ otherwise. Every person i has some private valuation $\theta_i \in \mathbb{R}$ for the project, positive or negative. Preferences are linear, so i's utility can be written as

$$u_i(x, \theta) = \theta_i k(\theta) - t_i(\theta).$$

Here $x = (k, t)$ stands for some direct mechanism which prescribes outcome $k(\theta)$ and payment profile $t(\theta)$ given profile of reports θ.

1. Calculate the efficient allocation rule $k^*(\theta)$.
2. Calculate the "efficient-ignoring-i" allocation rule $k^{-i}(\theta_{-i})$.
3. Calculate VCG transfers $t^{VCG}(\theta)$ that support k^*.
4. Is the mechanism ex post IR?
5. Is the mechanism ex post budget balanced?

Problem 3: Collusive mechanism in Cournot duopoly

Consider a Cournot duopoly with a inverse demand function $P(Q) = 1 - Q$ where Q is aggregate quantity. Suppose that each firm $i = 1, 2$ has constant marginal cost θ_i. This marginal cost is drawn uniformly from $[0, \frac{1}{2}]$. The realizations for the two firms are independent. Suppose that the firms observe their cost level, but not their rival's cost level prior to choosing their quantity.

Imagine that the two firms are able to collude by committing to a collusive "mechanism" whose outcomes are assignments of a quantity and a transfer payment to each of the two firms as a function of the announcements by the two firms of their cost type. Let $(k_i(\theta), t_i(\theta))$ denote the output level and transfer assigned to firm i if the announced profile of types (costs) is $\theta = (\theta_1, \theta_2)$.

1. We can use the VCG mechanism to implement the *profit*-maximizing production decisions in dominant strategies. Explain why. (Given that collusion is not usually perceived as an efficient outcome.)
2. Derive the VCG mechanism for this setting.
   (a) Find the profit-maximizing output profile $k^*(\theta)$.
   (b) Find the output profile $k^{-i}(\theta_{-i})$ that maximizes profit of firm $-i$ given its type θ_{-i}.
   (c) Find the VCG transfers and describe the VCG mechanism.
3. Argue why the firms would want the collusive mechanism to be or not be budget balanced and/or individually rational for the participants. If it should, argue which notions of IR and BB are the most reasonable to demand in this setting.
4. Is the VCG mechanism budget balanced? Is it individually rational assuming firms' outside options are zero (i.e., each firm's choice is between participating in the agreement and leaving the industry)?
5. Now suppose instead that either firm can reject the mechanism's prescription once it has been announced (at ex post stage), in which case firms go back to playing Cournot outcome. In which cases – i.e., for which realizations of (θ_1, θ_2) – would a firm want to back out of the agreement? Give formal conditions and explain them the best you can.

   (Assume that firms are not strategic about this contingency when making their reports to the mechanism, so truthful reporting is still an equilibrium.)

Problem 4: Spiteful exchange

Carl stole a coral from Clara; Clara stole Carl's clarinet. Once everything's been said and done, they are debating whether to exchange the stolen items back, $k \in \{0, 1\}$. Carl's own valuation for getting back the clarinet and returning the coral is given by θ_1, which is his private information. Clara's analogous valuation for returning the clarinet and recovering the coral is θ_2. Both players, however, are spiteful, so they want to maximize own value and minimize the other person's value. In the end, Carl's utility function u_1 and Clara's u_2 are given by

$$u_1(k, t, \theta) = \theta_1 k(\theta) - \alpha\theta_2 k(\theta) - t_1(\theta),$$
$$u_2(k, t, \theta) = \theta_2 k(\theta) - \alpha\theta_1 k(\theta) - t_2(\theta),$$

where α is the common animosity parameter, and t_i represent transfers to the mechanism.

Propose a welfare-maximizing mechanism (describe it fully and explain how you derived it) for each of the following cases:

1. $\alpha = 1$;

2. $\alpha \in (0, 1)$;
3. $\alpha > 1$.

Would the resulting mechanisms be individually rational and/or budget balanced?