Exercises for Lecture 6: Optimal mechanisms.

Problem 1: Malevolent Judicial Design

Revisit the Judicial Design exercise (L2 problem 4). Suppose that the defendant's type is $\theta \sim U[0,1]$, which stands for the probability of being acquitted if the defendant goes to the trial. Suppose that the DA wants to maximize the expected length of the defendant's sentence, i.e. to minimize the defendant's expected payoff. (So the DA gets a payoff of 1 for a life sentence and a payoff of r for a reduced sentence which would give the defendant a payoff of $-r$.)

1. Among the incentive-compatible mechanisms you identified, what is the optimal mechanism for the DA?
2. How does your answer change if going to trial imposes additional cost $c \in (0,1)$ on the DA (but not on the defendant) relative to agreeing on a plea bargain?

Problem 2: Optimal Procurement Mechanism

Consider an inverted setting to the one discussed in the lecture: a seller now has an item of some privately known quality $\theta \in [0,1]$. Here θ also equals the seller's valuation of the item, and the buyer's valuation is given by $v(\theta) > \theta$. The *buyer* designs a direct revelation mechanism (k,t) to purchase this item, where $k(\theta)$ is the probability of trade and $t(\theta)$ is the payment from the buyer to the seller.[1] Players' expected utilities given the seller's true type θ and the seller's report $\hat{\theta}$ are then given by

$$U_S(\hat{\theta}|\theta) = -\theta k(\hat{\theta}) + t(\hat{\theta}),$$
$$U_B(\hat{\theta}|\theta) = v(\theta)k(\hat{\theta}) - t(\hat{\theta}).$$

1. Show that for an allocation rule $k(\theta)$ to be IC for the seller, it must be weakly *decreasing* in θ.
2. ERP implies that the payoff of the seller of type θ can be written as $U_S(\theta|\theta) = U_S(1|1) + \int_\theta^1 k(s)ds$. Using this expression, show that the buyer's expected utility in any IC DRM (k,t) is given by:
$$\mathbb{E}_\theta[U_B(\theta|\theta)] = \int_0^1 k(\theta)VS(\theta)\phi(\theta)d\theta - U_S(1|1),$$
where $VS(\theta) = v(\theta) - \theta - \frac{\Phi(\theta)}{\phi(\theta)}$.
3. Explain each component of $VS(\theta)$.
4. Suppose from now on that $\theta \sim U[0,1]$. Find the optimal allocation rule when $v(\theta) = \frac{3\theta}{2}$ for $\theta \in [0, \frac{1}{3}]$ and $v(\theta) = \frac{5\theta}{2} - \frac{1}{3}$ for $\theta \in [\frac{1}{3}, 1]$.
5. Find the optimal allocation rule when $v(\theta) = \frac{5\theta}{2}$ for $\theta \in [0, \frac{1}{3}]$ and $v(\theta) = \frac{3\theta}{2} + \frac{1}{3}$ for $\theta \in [\frac{1}{3}, 1]$.

Problem 3: Uber Optimal Algorithm

Suppose you are the head economist of Uber and you are designing the economic side of the matching algorithm. Your goal is to pay the drivers as little as possible, while also ensuring their participation. In particular, consider the following situation: some consumer has placed a fare (a ride order) in the app, and

[1] Note that this formulation of the problem explicitly constrains the mechanism to be ex post budget balance requirement.

there is a driver available in the vicinity. You (the app) are then effectively bargaining with the driver for how little money they are willing to accept in order to complete this fare.

Suppose the app charges the consumer some amount w for the ride. This w is fixed and commonly known by all players, including the driver and the designer. The driver values their time at $\theta \sim U[0,1]$, which is their private information. You are designing a direct revelation mechanism $\{k(\theta), p(\theta)\}_{\theta\in[0,1]}$, which works as follows:

(i) the driver reports θ to the app;

(ii) the app offers the fare to the driver with probability $k(\theta) \in [0,1]$;[2]

(iii) the driver can accept or reject the fare;

(iv) if the driver accepted and completed the fare, they receive payment $p(\theta)$;

(v) if the driver declined or was not offered the fare, they receive utility θ.

Therefore, the driver's expected utility from receiving with probability k a fare that pays p is

$$u(k,t,\theta) = k \cdot p + (1-k) \cdot \theta,$$

and their outside option (from rejecting the fare or the whole mechanism) is $\underline{U}(\theta) = \theta$.

Your task is to devise an optimal mechanism (k, p) that maximizes the firm's expected revenue $\mathbb{E}_\theta[k(\theta) \cdot (w - p(\theta))]$ subject to the driver's standard IC constraint and interim IR constraint $u(k,t,\theta) \geq \underline{U}(\theta)$. To derive this mechanism, follow the steps below.

1. Show that in any IC mechanism, $k(\theta)$ must be weakly decreasing.
2. Show that in any IC mechanism, the following holds for all θ (where $U(\theta) \equiv k(\theta)p(\theta) + (1-k(\theta))\theta$):

$$U(\theta) = U(1) - \int_\theta^1 (1-k(s))ds. \tag{1}$$

3. Show that in any IC mechanism, the firms' expected profit (from this driver) can be expressed in the following way as a function of the allocation rule k and $U(1)$:

$$\mathbb{E}_\theta[k(\theta) \cdot (w - p(\theta))] = \int_0^1 [2\theta + k(\theta) \cdot (w - 2\theta)]\, d\theta - U(1). \tag{2}$$

4. Find the allocation rule k that maximizes the expected profit (2). Does it satisfy the monotonicity requirement from part 1? If not, what is the optimal monotone allocation rule?
5. Argue (formally if you can) why type $\theta = 1$ is the one for whom the IR constraint will be the most binding. I.e., show that if IR holds for $\theta = 1$ then it also holds for all other $\theta \in [0, 1)$.
6. Suppose $w = 1$. Derive the payment rule $p(\theta)$ that supports the optimal [monotone] allocation rule.
7. Suppose $w = 2$. Derive the payment rule $p(\theta)$ that supports the optimal [monotone] allocation rule.
8. Are the mechanisms you obtained in the two previous questions ex post IR for the drivers? I.e., will the driver always accept any fare that is offered to them?

[2] Think that if the fare is not offered to the driver, then the consumer sees an empty screen, as if there were no available drivers in their area.

Problem 4: Divine intervention

The year is 854 AD. The place is Denmark. The reigning king Horik is challenged by his nephew Guttorm for the claim to the kingdom. Both know that a grand battle between them in inevitable, and both are praying to Odin and the rest of Æsir to tilt the outcome of this battle in their favor. You are to assume the role of Odin and to decide the outcome of the battle.

In particular, you are a mechanism designer dealing with two players $i = H, G$. Every player i has a private type $\theta_i \sim U[0, 1]$ (θ_H and θ_G are indepedent). An outcome is given by $x = (k, t)$, where $k = (k_H, k_G)$ is an allocation such that $k_i \in [-1, 1]$ and $k_H + k_G \leq 0$, and t is a vector of transfers. Players' payoffs are given by $u_i(x, \theta) = k_i\theta_i - t_i$. The outside options are normalized to zero.

In our story, allocations represent who wins the battle: e.g., $(k_H, k_G) = (1, -1)$ means Horik wins while Guttorm loses and is killed in battle. Restriction $k_H + k_G \leq 0$ means that they cannot both win, but can both lose. Transfer t_i need not mean money, but rather (the negative of) favor that the gods will show to i outside of battle, in life or afterlife. Horik's and Guttorm's types represent their paltriness. Higher θ_i means that i would very much prefer to become a king and is very afraid of death. Low θ_i means that i values honorable death in combat almost as highly as reign over the realm. Finally, the "outside option" represents fleeing from the battle, retaining life but losing the kingdom. The level of "favor" t_i in this case is normalized to zero (i.e., favor is measured relative to this scenario).

For the first part of this problem, suppose that Odin & the co-gods love a good battle (so do not want players to flee), but are otherwise benevolent and do not care about favor. They thus decide to use the generalized VCG mechanism to determine the outcome.

1. Find the efficient allocation rule $k^*(\theta) = \arg\max_k\{u_H(k, \theta) + u_G(k, \theta)\}$.
2. Find the least charitable types $\tilde{\theta}_i$.
3. Calculate the gVCG transfer rule $t^{gVCG}(\theta)$ that supports the efficient allocation rule.

Suppose now instead that the gods are not benevolent, but try to minimize the amount of favor they owe to mortals – i.e., to maximize $\mathbb{E}[t_H + t_G]$. That said, they still like to see a good battle, so ensuring that neither player decides to flee takes priority.

4. Find the optimal BIC mechanism that maximizes the expected "revenue" $\mathbb{E}_\theta[t_H(\theta) + t_G(\theta)]$ subject to the players' IR constraints.