

# Mechanism Design

1: Definitions, Implementation

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# This slide deck:

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- 1 Defining a Mechanism
- 2 Revelation Principle
- 3 Dominant Strategy Implementation
- 4 Bayesian Implementation

# What is a mechanism?

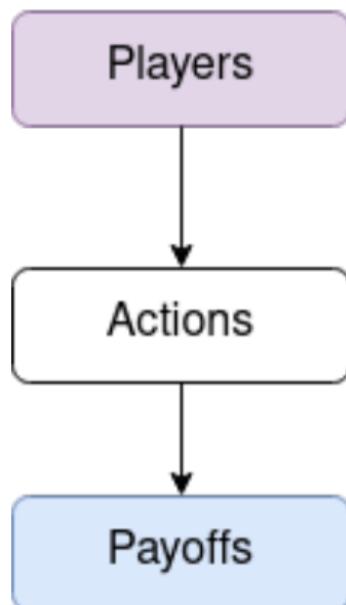
Let's reverse engineer from a simpler question: **What is a game?**

- 1 Set of players  $i \in \{1, \dots, N\}$
- 2 Set of actions  $A_i$  for every  $i$ ; set of action profiles  $A \equiv \times_{i \in N} A_i$
- 3 Collection of utility functions  $u_i : A \rightarrow \mathbb{R}$

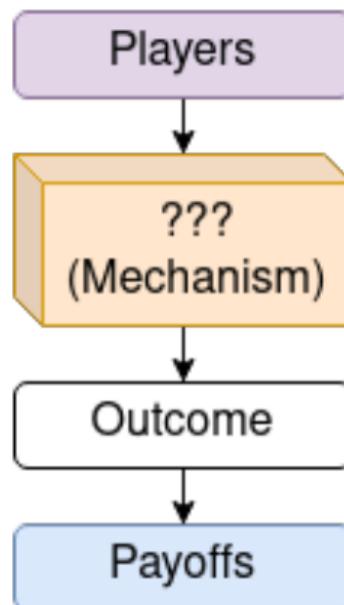
(This is a *normal-form game*. All extensive-form games ("trees") and incomplete-information games can be represented as normal-form games.)

Which parts of this definition are fixed at a higher level, and which can we *design* as a part of a *mechanism*?

## Game



## Mechanism



# General Problem Set-up

In our MD problem, the following environment will be **fixed**:

- $N$  agents,
- set  $X$  of **outcomes**,
- each agent  $i$  has **type**  $\theta_i \in \Theta_i$ :
  - describes agent's **information**,
  - describes agent's **preferences**;
- the type profile  $\theta = (\theta_1, \dots, \theta_N)$  is distributed according to a distribution  $F$  with p.d.f.  $\phi$ ,
  - (often a missing subscript denotes a vector of respective objects)
  - distribution  $F$  is commonly known and agreed upon
- each agent has a **utility** function  $u_i(x, \theta_i)$  that depends on the collective choice  $x \in X$  and his type  $\theta_i$ ,

# Mechanism

- a mechanism is a game played by the agents
- each agent has an action set  $A_i$  in this game

## Definition (mechanism)

A **mechanism**  $\Gamma = (A_1, \dots, A_N, g(\cdot))$  is a collection of:

- $N$  **strategy sets**  $(A_1, \dots, A_N)$  and
- an **outcome function**  $g : A_1 \times \dots \times A_N \rightarrow X$ .

# Social Choice Function

## Definition (Social choice function)

A **social choice function** is a function  $f : \Theta_1 \times \dots \times \Theta_N \rightarrow X$  that assigns to each profile of types  $(\theta_1, \dots, \theta_N)$  a collective choice  $f(\theta_1, \dots, \theta_N) \in X$ .

- gives a desired outcome as a function of the agents' types

# Implementation

## Definition (implementation)

Mechanism  $\Gamma = (A_1, \dots, A_N, g(\cdot))$  **implements** the s.c.f.  $f$  if there is **an equilibrium** strategy profile  $(a_1^*, \dots, a_N^*)$  of the Bayesian game induced by  $\Gamma$  **such that**

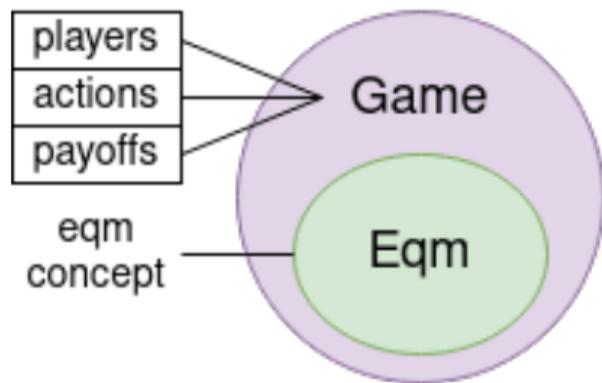
$$g(a_1^*(\theta_1), \dots, a_N^*(\theta_N)) = f(\theta_1, \dots, \theta_N)$$

for all  $(\theta_1, \dots, \theta_N) \in \Theta_1 \times \dots \times \Theta_N$ .

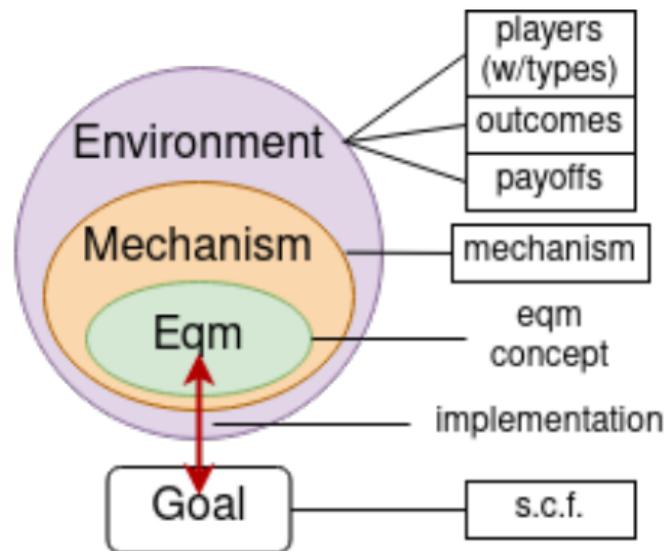
# Summary of definitions

- S.c.f.  $f$  describes what we want to achieve;
- Mechanism  $\Gamma = (S, g)$  describes what we do and how;
- Implementability says whether we have achieved our goal.

## Full problem setup



**Game Theory  
problem**



**Mechanism Design  
problem**

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# Revelation Principle

- Main cheat in Mechanism Design! No need to bruteforce through uncountable numbers of different games! It is enough to just... (click to see more)

# Revelation Principle

- Main cheat in Mechanism Design! No need to bruteforce through uncountable numbers of different games! It is enough to just...
- Instead of making players play the game, ask them for their  $\theta_i$  and promise to play on their behalf!
- Requires that the designer has commitment power.
  - Strong assumption, sometimes reasonable(?)
  - The necessary evil for our purposes.
  - Useful for formal analysis, but in the end the resulting mechanism can *sometimes* work even without commitment.

# Revelation Principle: Definitions

Fix some s.c.f.  $f : \Theta \rightarrow X$ .

## Definition (Direct revelation mechanism)

A **direct revelation mechanism** for  $f$  is a mechanism in which  $A_i = \Theta_i$  for all  $i$  and  $g(\theta) = f(\theta)$

## Definition (Truthful implementation)

S.c.f.  $f$  is **truthfully implementable** if it can be implemented by a direct revelation mechanism.

# Revelation Principle: Statement

## Revelation principle (blanket statement)

Suppose there exists a mechanism  $\Gamma = (A_1, \dots, A_N, g)$  that implements the social choice function  $f$ .

Then  $f$  is **truthfully implementable**.

- The “theorem” above is informal.
  - “Implementation” requires “an equilibrium”, which can mean a million different things.
  - We will now plug in some specific equilibrium concepts.

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# Game Theory Recap: Dominant Strategy

- strategy  $a_i$  is a full contingent plan of play
- strategy  $a_i$  is **dominant** for agent  $i$  if it is best *no matter what the other players do*

## Definition (dominant strategy)

Given mechanism  $\Gamma = (A, g)$ ,  $a_i : \Theta_i \rightarrow A_i$  is a **dominant strategy** if for all  $\theta_i \in \Theta_i$

$$u_i(g(a_i(\theta_i), a_{-i}), \theta_i) \geq u_i(g(\hat{a}_i, a_{-i}), \theta_i)$$

for all  $\hat{a}_i \in A_i$  and all  $a_{-i} \in A_{-i}$ .

- our definition slightly different from the standard – does not require strict inequality

# Dominant Strategy Equilibrium

- in a **dominant strategy equilibrium** every player plays a dominant strategy

## Definition (dominant strategy equilibrium)

A strategy profile  $(a_1^*, \dots, a_N^*)$  is a **dominant strategy equilibrium** of mechanism  $\Gamma = (A_1, \dots, A_N, g)$  if for all  $i$  and all  $\theta_i \in \Theta_i$

$$u_i(g(a_i^*(\theta_i), a_{-i}), \theta_i) \geq u_i(g(\hat{a}_i, a_{-i}), \theta_i)$$

for all  $\hat{a}_i \in A_i$  and all  $a_{-i} \in A_{-i}$ .

Now let's finally be formal about all our definitions.

# Dominant Strategy Implementation

- A mechanism **implements  $f$  in dominant strategies** if
  - the game induced by the mechanism has a dominant strategy equilibrium
  - the outcome in this equilibrium coincides with  $f$

## Definition (implementation in dominant strategies)

A mechanism  $\Gamma = (A_1, \dots, A_N, g)$  **implements** the social choice function  $f$  **in dominant strategies** if there exists a dominant strategy equilibrium  $(a_1^*, \dots, a_N^*)$  of  $\Gamma$  such that  $g(a_1^*(\theta_1), \dots, a_N^*(\theta_N)) = f(\theta)$  for all  $\theta \in \Theta$ .

# Good Implementation Concept?

- very robust equilibrium concept
  - no need to predict what the other players will play
  - no need to know the type distribution  $\phi$
  - works even if
    - players don't know  $\phi$  or even if players believe in different  $\phi_i$  (protects from players' model misspecification)
    - players think that other players are not rational
- not a panacea
  - does not rule out other weird Nash Equilibria (example: second-price auction)
  - is not necessarily collusion-proof
  - does not protect from designer's model misspecification

Bottom line: it's as good as they get, but far from perfect.

# Dominant Strategy Incentive Compatibility

## Theorem (Revelation Principle for Dominant Strategies)

*Suppose there exists a mechanism  $\Gamma = (A_1, \dots, A_N, g)$  that implements the social choice function  $f$  in dominant strategies.*

*Then  $f$  is **truthfully implementable** in dominant strategies.*

## Definition (Dominant Strategy Incentive Compatibility)

*" $f$  is **dominant strategy incentive compatible** (DSIC)"*

*means the exact same thing as*

*" $f$  is **truthfully implementable in dominant strategies**".*

## DS Revelation Principle: Proof

Let  $\Gamma$  implement  $f$  in dominant strategies, i.e. there is a strategy profile  $(a_1^*, \dots, a_N^*)$  such that  $g(a_1^*(\theta_1), \dots, a_N^*(\theta_N)) = f(\theta)$  for all  $\theta$ , and for all  $i$  and  $\theta_i \in \Theta_i$ ,

$$u_i(g(a_i^*(\theta_i), a_{-i}), \theta_i) \geq u_i(g(\hat{a}_i, a_{-i}), \theta_i)$$

for all  $\hat{a}_i \in A_i$  and all  $a_{-i} \in A_{-i}$ .

Then

$$u_i(g(a_i^*(\theta_i), a_{-i}^*(\theta_{-i})), \theta_i) \geq u_i(g(a_i^*(\hat{\theta}_i), a_{-i}^*(\theta_{-i})), \theta_i)$$

for all  $\hat{\theta}_i \in \Theta_i$ ,  $\theta_{-i} \in \Theta_{-i}$ .

Since  $g(a^*(\theta)) = f(\theta)$ ,

$$u_i(f(\theta_i, \hat{\theta}_{-i}), \theta_i) \geq u_i(f(\hat{\theta}_i, \hat{\theta}_{-i}), \theta_i)$$

for all  $\hat{\theta}_{-i} \in \Theta_{-i}$ .

# Revelation Principle: Is it cool or is it cool?

- Idea: to solve the problem **mathematically**, it is enough to only look at **direct mechanisms**!
  - This result allows to quickly check whether a given  $f$  is [DS] implementable.
  - If yes, gives you a mechanism to implement it.
  - If not, helps you describe a set of implementable s.c.f. and pick second best.
  - *Yours today for ~~only \$49.99 + shipping~~ FREE with a qualifying Mechanism Design course!*
- Translating that solution to **the real world** may (and often does) result in an **indirect mechanism**! We'll see some examples.

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# DSIC vs BIC

- We've looked at DS implementation so far. Robust but demanding. Can we get more mileage by relaxing the equilibrium notion?
- Now: use standard **Bayes-Nash Equilibrium** as solution concept. Weaker equilibrium concept, so:
  - we are less confident it will produce the intended outcome, but
  - it can implement more(?) s.c.f.-ns.
  - (there's a literature studying whether sets of DSIC and BIC s.c.f.-ns are equal in special settings)

# Bayesian Implementation

Start with the **general model** as before:

- $N$  agents;
- set of alternatives  $X$ ;
- type  $\theta_i \in \Theta_i$  is private information of  $i$ ;
- common prior belief  $\phi \in \Delta(\Theta)$  about distribution of types;
- utility functions  $u_i(x, \theta_i)$ ;
- each agent uses Bayes' rule to form a belief over other agents' types

$$\phi(\theta_{-i}|\theta_i) = \phi(\theta_i, \theta_{-i}|\theta_i) = \frac{\phi(\theta_i, \theta_{-i})}{\int_{\tilde{\theta}_{-i} \in \Theta_{-i}} \phi(\theta_i, \tilde{\theta}_{-i}) d\tilde{\theta}_{-i}}.$$

# Bayes-Nash Equilibrium

## Definition (Bayes-Nash equilibrium)

The **strategy profile**  $a^* = (a_1^*, \dots, a_N^*)$  with  $a_i^* : \Theta_i \rightarrow A_i$  is a **Bayes-Nash equilibrium** of the mechanism  $\Gamma = (A_1, \dots, A_N, g)$  if, for all  $i$  and all  $\theta_i \in \Theta_i$ ,

$$\mathbb{E}_{\theta_{-i}} [u_i(g(a_i^*(\theta_i), a_{-i}^*(\theta_{-i})), \theta_i) | \theta_i] \geq \mathbb{E}_{\theta_{-i}} [u_i(g(\hat{a}_i, a_{-i}^*(\theta_{-i})), \theta_i) | \theta_i]$$

for all  $\hat{a}_i \in A_i$ .

# Bayes-Nash Equilibrium

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for all  $\hat{a}_i \in A_i$ .

- Standard NE reasoning: if everyone else plays eqm strats,  $i$  has no incentive to deviate.
- (This definition is for pure strategies, but there is no problem in allowing mixed strategies.)
- Expectations are taken w.r.t. distribution  $F(\theta)$

# Bayesian Implementation

## Definition (Bayesian implementation)

**Mechanism**  $\Gamma = (A_1, \dots, A_N, g)$  implements s.c.f.  $f$  in Bayes-Nash equilibrium if there is a BNE  $a^* = (a_1^*, \dots, a_N^*)$  of  $\Gamma$  such that  $f(\theta) = g(a^*(\theta))$  for all  $\theta \in \Theta$ .

# Bayesian Implementation

## Definition (Bayesian implementation)

**Mechanism**  $\Gamma = (A_1, \dots, A_N, g)$  **implements s.c.f.  $f$  in Bayes-Nash equilibrium** if there is a BNE  $a^* = (a_1^*, \dots, a_N^*)$  of  $\Gamma$  such that  $f(\theta) = g(a^*(\theta))$  for all  $\theta \in \Theta$ .

## Definition (Bayesian implementability)

**S.c.f.  $f$  is implementable in BNE** if there exists  $\Gamma$  which implements it in BNE.

# Truthful Bayesian Implementation

## Definition (Truthful Bayesian implementation)

S.c.f.  $f$  is **truthfully implementable in BNE** (=Bayesian Incentive Compatible, **BIC**) if  $a_i^*(\theta_i) = \theta_i$  is a BNE of the direct revelation mechanism  $\Gamma = (\Theta_1, \dots, \Theta_N, f)$ .

That is, for all  $i, \theta_i$ , and  $\hat{\theta}_i \in \Theta_i$ ,

$$\mathbb{E}_{\theta_{-i}} [u_i(f(\theta_i, \theta_{-i}), \theta_i) | \theta_i] \geq \mathbb{E}_{\theta_{-i}} [u_i(f(\hat{\theta}_i, \theta_{-i}), \theta_i) | \theta_i].$$

Every player is asked for their type; reporting truthfully is a BNE.

# Revelation principle

## Theorem (Revelation principle for Bayes-Nash equilibrium)

*If there exists a mechanism  $\Gamma = (A_1, \dots, A_N, g)$  that implements  $f$  in BNE, then  $f$  is truthfully implementable in BNE.*

The proof is pretty much the same as for DSIC.