Exercises for Lecture 5: gVCG, AGV.

Problem 1: Efficient public good provision 3

Consider problem 2 from L3 problem set.

Assume now that players' valuations are distributed according to $\theta_i \sim U[-\hat{\theta}, \hat{\theta}]$ for all i, and that the public project has some known social cost $c > 0$. All players' outside options are zero: $\underline{U}_i(\theta_i) = 0$.

Derive the gVCG transfers.

Problem 2: AGV and public goods

Consider the public good provision problem (again). Suppose now that there are only two individuals: $i = 1, 2$, their valuations for the public project are $\theta_i \sim i.i.d. U[-1, 1]$, and the cost is $c \in [0, 1]$ (known to all agents).

1. Calculate the AGV transfers for this problem.
2. Do the players' payments to the mechanism cover the project cost c if and only if the project is implemented? (I.e., is the mechanism exactly budget balanced once we account for the project costs?)

Problem 3: Myerson-Satterthwaite theorem

Consider the **bilateral trade** problem discussed in class: one buyer, one seller, one item. The seller's valuation for the item is given by his private type $\theta_S \sim U[0, 1]$, and the buyer's valuation is given by his private type $\theta_B \sim U[0, 1]$, independent of θ_S. The outside options are given by $\underline{U}_S(\theta_S) = \theta_S$ and $\underline{U}_B(\theta_B) = 0$ respectively. The utilities of the two players are Euclidean and are given by:

$$u_S = v(k, \theta_S) - t_S(\theta) = \theta_S(1 - k) - t_S(\theta)$$
$$u_B = v(k, \theta_B) - t_B(\theta) = \theta_B k - t_B(\theta)$$

where $k(\theta) \in [0, 1]$ is the probability of trade given type profile θ. The designer would like to create an efficient market for these players (i.e., implement the efficient allocation rule)

Derive the gVCG transfers for this problem. Show that the resulting mechanism is not ex ante budget balanced (not even weakly).

Problem 4: Auction with non-trivial seller valuation

A seller, $i = 1$, possesses a single indivisible object for which there are two potential buyers. Each buyer $i \in \{2, 3\}$ has value v_i for the good and the seller has an opportunity cost c from selling the good. Utility is quasi-linear in money, so if buyer i purchases the good at price p, his final utility is $v_i - p$, and the seller's utility is $p - c$. Each agent has zero utility if he does not trade and zero is therefore also the reservation utility of each agent.

Each v_i is drawn independently from the same distribution F which has full support on $[0, \bar{v}]$, and c is drawn independently from a distribution G which has support $[0, \bar{c}]$. Assume that F and G satisfy all of the conditions necessary for the revenue equivalence theorem and our characterization results in class.

A mechanism consists of two collections of functions, $q(c, v_2, v_3)$ and $t(c, v_2, v_3)$, where $q(c, v_2, v_3)$ is a probability distribution that prescribes the probabilities that the good will be allocated to each of the three agents, and $t(c, v_2, v_3)$ gives the list of transfers paid by each of the three agents. Say that a mechanism is *feasible* if it is Bayesian incentive compatible, interim individually rational, and ex ante exactly budget-balanced.

1. What is the efficient allocation rule?
2. Assume $\bar{c} = \bar{v}$. Show that there does not exist a feasible mechanism that implements the efficient allocation rule. *Hint: use gVCG and its properties.*
3. Now assume $\bar{v} > \bar{c}$. Show that the following is a sufficient condition for the existence of a feasible mechanism that implements the efficient allocation rule:

$$\mathbb{E}(\min\{v_2, v_3\}) \geq \bar{c}.$$

4. Again assume $\bar{v} > \bar{c}$, but now suppose that there are N potential buyers with values drawn independently from F. Prove that for any F and G there is a $\bar{N}$ such that whenever $N > \bar{N}$ there exists a feasible mechanism that implements the efficient allocation rule.