Exercises for Lecture 5: gVCG, AGV.

Problem 1: Efficient public good provision 3

Consider problem 2 from L3 problem set.

Assume now that players' valuations are distributed according to $\theta_i \sim U[-\hat{\theta}, \hat{\theta}]$ for all i, and that the public project has some known social cost $c > 0$. All players' outside options are zero: $\underline{U}_i(\theta_i) = 0$.

Derive the gVCG transfers.

Solution

LCT for any i is $\tilde{\theta}_i = -\hat{\theta}$ (you do not actually need to calculate the expectation to find it, since the expression that $\tilde{\theta}_i$ minimizes is weakly monotone in θ_i – i.e., one of the edges of the support is the solution). The gVCG transfers are then given by

$$\begin{aligned}
t_i^{gVCG}(\theta) &= \max\left\{0, \sum_{j\neq i}\theta_j - \hat{\theta} - c\right\} - \left(\sum_{j\neq i}\theta_j - c\right)\cdot \mathbb{I}\left\{\sum_{j=1}^{N}\theta_j - c > 0\right\} \\
&= \begin{cases} 0 & \text{if } \sum_{j=1}^{N}\theta_j - c \leq 0, \\ -\left(\sum_{j\neq i}\theta_j - c\right) & \text{if } \sum_{j\neq i}\theta_j - \hat{\theta} - c \leq 0 < \sum_{j=1}^{N}\theta_j - c, \\ -\hat{\theta} & \text{if } \sum_{j\neq i}\theta_j - \hat{\theta} - c > 0. \end{cases}
\end{aligned}$$

Problem 2: AGV and public goods

Consider the public good provision problem (again). Suppose now that there are only two individuals: $i = 1, 2$, their valuations for the public project are $\theta_i \sim i.i.d. U[-1, 1]$, and the cost is $c \in [0, 1]$ (known to all agents).

1. Calculate the AGV transfers for this problem.
2. Do the players' payments to the mechanism cover the project cost c if and only if the project is implemented? (I.e., is the mechanism exactly budget balanced once we account for the project costs?)

Solution

Note: the lectures are somewhat vague, so you can get somewhat different expressions in part 1 depending on how exactly you approach the problem. The answer to part 2 should, however, be qualitatively the same in all of those cases.

The $\tilde{t}$ are as follows: for $i = 1, 2$,

$$\begin{aligned}
\tilde{t}_i(\theta_i) &= \mathbb{E}_{\theta_j}\left[(\theta_j - c)\cdot \mathbb{I}\{\theta_1 + \theta_2 \geq c\}\right] \\
&= \int_{\min\{c-\theta_i, 1\}}^{1} (\theta_j - c)\frac{1}{2}d\theta_j \\
&= \frac{1}{4}\left[(c-1)^2 - \theta_i^2\right]\cdot \mathbb{I}\{\theta_i \geq c - 1\};
\end{aligned}$$

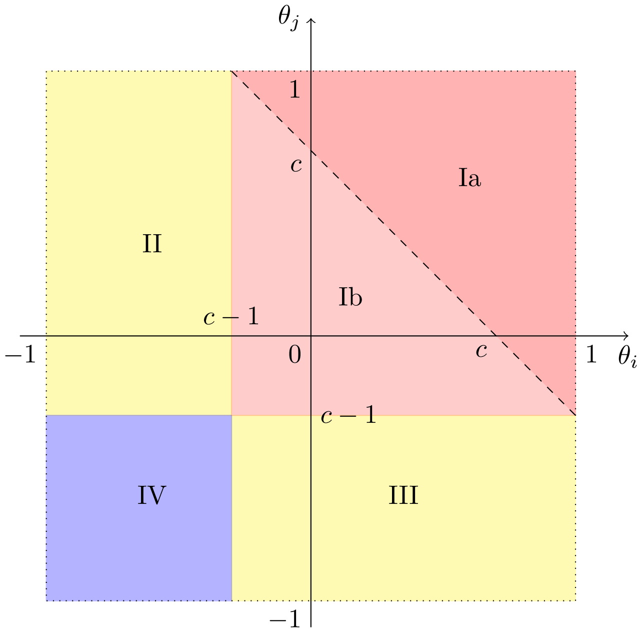

Figure 1: regions for AGV transfers

and the actual transfers are then $t_i^{AGV}(\theta) = \tilde{t}_j(\theta_j) - \tilde{t}_i(\theta_i)$ for $i = 1, 2$, which evaluates to

$$t_i^{AGV}(\theta) = \begin{cases} \frac{\theta_i^2 - \theta_j^2}{4} & \text{if } \theta_i, \theta_j \geq c-1 \text{ (region I)}, \\ \frac{(c-1)^2 - \theta_j^2}{4} & \text{if } \theta_j \geq c-1 > \theta_i \text{ (region II)}, \\ \frac{\theta_i^2 - (c-1)^2}{4} & \text{if } \theta_i \geq c-1 > \theta_j \text{ (region III)}, \\ 0 & \text{if } c-1 > \theta_i, \theta_j \text{ (region IV)}. \end{cases}$$

The regions are plotted in Figure 1. Note that the public project is implemented only in region Ia, but the two agents' transfers sum up to zero there, same as in all other regions. Therefore, the AGV mechanism does not cover the project cost. At this point you might think that this is because you need to compute $\tilde{t}_0$ and somehow include it in the players' payments. However, regardless of how you split this $\tilde{t}_0$ across agents, payments in regions Ia and Ib will always be continuous at the border (since $\tilde{t}_0$ would just add some constant to agents' payments), whereas to cover the cost of the project exactly, the sum of payments must be larger by c in Ia than in Ib.

Problem 3: Myerson-Satterthwaite theorem

Consider the **bilateral trade** problem discussed in class: one buyer, one seller, one item. The seller's valuation for the item is given by his private type $\theta_S \sim U[0,1]$, and the buyer's valuation is given by his private type $\theta_B \sim U[0,1]$, independent of θ_S. The outside options are given by $\underline{U}_S(\theta_S) = \theta_S$ and $\underline{U}_B(\theta_B) = 0$ respectively. The utilities of the two players are Euclidean and are given by:

$$\begin{aligned} u_S &= v(k, \theta_S) - t_S(\theta) = \theta_S(1-k) - t_S(\theta) \\ u_B &= v(k, \theta_B) - t_B(\theta) = \theta_B k - t_B(\theta) \end{aligned}$$

where $k(\theta) \in [0,1]$ is the probability of trade given type profile θ. The designer would like to create an efficient market for these players (i.e., implement the efficient allocation rule)

Derive the gVCG transfers for this problem. Show that the resulting mechanism is not ex ante budget balanced (not even weakly).

Solution

It is straightforward to see that the efficient allocation rule k^* is given by:

$$k^* = \begin{cases} 1 & \text{if } \theta_S \leq \theta_B \\ 0 & \text{if } \theta_S > \theta_B \end{cases}$$

Our next step is to construct the gVCG transfers that implement the efficient allocation. They are given by:

$$\begin{aligned} t_S^{gVCG} &= v_B(k^*(\tilde{\theta}_S, \theta_B), \theta_B) + v_S(k^*(\tilde{\theta}_S, \theta_B), \tilde{\theta}_S) - \\ &\quad - v_B(k^*(\theta_S, \theta_B), \theta_B) - \underline{U}_S(\tilde{\theta}_S) \\ t_B^{gVCG} &= v_S(k^*(\tilde{\theta}_B, \theta_S), \theta_S) + v_B(k^*(\tilde{\theta}_B, \theta_S), \tilde{\theta}_B) - \\ &\quad - v_S(k^*(\theta_B, \theta_S), \theta_S) - \underline{U}_B(\tilde{\theta}_B) \end{aligned}$$

Noticing that $v_B(k^*(\theta)) + v_S(k^*(\theta)) = \max\{\theta_B, \theta_S\}$ and plugging in the efficient allocation k^* and the outside options $\underline{U}_i$, we get

$$\begin{aligned} t_S^{gVCG} &= \max\{\tilde{\theta}_S, \theta_B\} - \theta_B k^*(\theta_S, \theta_B) - \tilde{\theta}_S \\ t_B^{gVCG} &= \max\{\tilde{\theta}_B, \theta_S\} - \theta_S (1 - k^*(\theta_S, \theta_B)) \end{aligned}$$

The least charitable types $\tilde{\theta}_i$ of each player are defined as:

$$\begin{aligned} &\tilde{\theta}_i \in \arg\min_{\theta_i \in \Theta_i} \left\{ \mathbb{E}_{\theta_{-i}} \left[v_B(k^*(\theta_i, \theta_{-i}), \theta_j) + v_S(k^*(\theta_i, \theta_{-i}), \theta_j) - \underline{U}_i(\theta_i) \right] \right\} \\ \Rightarrow\ &\tilde{\theta}_B \in \arg\min_{\theta_B \in [0,1]} \left\{ \mathbb{E}_{\theta_S} \left[\max\{\theta_B, \theta_S\} \right] \right\} = \arg\min_{\theta_B \in [0,1]} \left\{ \int_0^1 \max\{\theta_B, \theta_S\} \phi(\theta_S) d\theta_S \right\} \\ &= \arg\min_{\theta_B \in [0,1]} \left\{ \int_0^{\theta_B} \theta_B d\theta_S + \int_{\theta_B}^1 \theta_S d\theta_S \right\} = \arg\min_{\theta_B \in [0,1]} \left\{ \theta_B^2 + \frac{1 - \theta_B^2}{2} \right\} \\ &= \{0\}; \\ &\tilde{\theta}_S \in \arg\min_{\theta_S \in [0,1]} \left\{ \mathbb{E}_{\theta_B} \left[\max\{\theta_B, \theta_S\} - \theta_S \right] \right\} = \arg\min_{\theta_S \in [0,1]} \left\{ \frac{1}{2} + \frac{\theta_S^2}{2} - \theta_S \right\} = \{1\}. \end{aligned}$$

So in the end we have $\tilde{\theta}_B = 0$, $\tilde{\theta}_S = 1$. Plugging these into the respective expressions for transfers, we get the following (because $\max\{\tilde{\theta}_S, \theta_B\} = \max\{1, \theta_B\} = 1$, and $\max\{\tilde{\theta}_B, \theta_S\} = \max\{0, \theta_S\} = \theta_S$):

$$\begin{aligned} t_S^{gVCG} &= -\theta_B k^*(\theta_S, \theta_B) \\ t_B^{gVCG} &= \theta_S k^*(\theta_S, \theta_B) \end{aligned}$$

Recall that BB (budget balance) is defined as $t_S + t_B \geq 0$. The sum of gVCG transfers is:

$$t_S + t_B = \begin{cases} \theta_S - \theta_B < 0 & \text{if } \theta_S \leq \theta_B, \\ 0 & \text{if } \theta_S > \theta_B. \end{cases}$$

Hence, we have now shown that the gVCG mechanism is not budget balanced (ex ante or ex post). However, it is the mechanism that yields the highest expected revenue $\mathbb{E}_\theta\left(t_S(\theta) + t_B(\theta)\right)$ among all mechanisms that are efficient, BIC, and interim IR. Therefore, there does not exist a mechanism for the bilateral trade problem which is efficient, BIC, interim IR, and ex ante BB.

Problem 4: Auction with non-trivial seller valuation

A seller, $i = 1$, possesses a single indivisible object for which there are two potential buyers. Each buyer $i \in \{2, 3\}$ has value v_i for the good and the seller has an opportunity cost c from selling the good. Utility is quasi-linear in money, so if buyer i purchases the good at price p, his final utility is $v_i - p$, and the seller's utility is $p - c$. Each agent has zero utility if he does not trade and zero is therefore also the reservation utility of each agent.

Each v_i is drawn independently from the same distribution F which has full support on $[0, \bar{v}]$, and c is drawn independently from a distribution G which has support $[0, \bar{c}]$. Assume that F and G satisfy all of the conditions necessary for the revenue equivalence theorem and our characterization results in class.

A mechanism consists of two collections of functions, $q(c, v_2, v_3)$ and $t(c, v_2, v_3)$, where $q(c, v_2, v_3)$ is a probability distribution that prescribes the probabilities that the good will be allocated to each of the three agents, and $t(c, v_2, v_3)$ gives the list of transfers paid by each of the three agents. Say that a mechanism is *feasible* if it is Bayesian incentive compatible, interim individually rational, and ex ante exactly budget-balanced.

1. What is the efficient allocation rule?
2. Assume $\bar{c} = \bar{v}$. Show that there does not exist a feasible mechanism that implements the efficient allocation rule. *Hint: use gVCG and its properties.*
3. Now assume $\bar{v} > \bar{c}$. Show that the following is a sufficient condition for the existence of a feasible mechanism that implements the efficient allocation rule:

$$\mathbb{E}(\min\{v_2, v_3\}) \geq \bar{c}.$$

4. Again assume $\bar{v} > \bar{c}$, but now suppose that there are N potential buyers with values drawn independently from F. Prove that for any F and G there is a $\bar{N}$ such that whenever $N > \bar{N}$ there exists a feasible mechanism that implements the efficient allocation rule.

Solution

Note: this problem has not been brought in conformance with the notation we use in class (allocations are q instead of k, valuations are v_i instead of θ_i, conflicting with our notation for real utilities $v_i(k, \theta_i)$, it mentions both prices p and transfers t...) This adds an extra layer of complexity that tests your skills in parsing text and context, rather than notation.

This solution uses the following notation: $a \vee b \equiv \max\{a, b\}$; $a \wedge b \equiv \min\{a, b\}$.

1. The efficient allocations must satisfy

$$q(c, v_2, v_3) = \begin{cases} (1,0,0) & \text{if } c > v_2, v_3, \\ (0,1,0) & \text{if } v_2 > c, v_3, \\ (0,0,1) & \text{if } v_3 > c, v_2. \end{cases}$$

   Ties can be broken arbitrarily.

2. Let's use the Krishna–Perry theorem. The least charitable types are 0 for the buyers and $\bar{c}$ for the seller (see problem 3).[1] The gVCG mechanism has the following transfers:

   - when no trade occurs, there are no transfers
   - when trade occurs between the seller and buyer i, i pays $v_{-i} \vee c$, the seller receives $v_i \wedge \bar{c}$, and $-i$ has transfer zero.

   If $\bar{v} = \bar{c}$ then $v_i \wedge \bar{c} = v_i$, so this mechanism yields weakly negative revenue:

$$\sum_i t_i(\theta) = \begin{cases} 0 & \text{if } c > v_2, v_3 \\ v_3 \vee c - v_2 < 0 & \text{if } v_2 > c, v_3 \\ v_2 \vee c - v_3 < 0 & \text{if } v_3 > c, v_2. \end{cases}$$

   Since gVCG mechanism has maximal revenue among efficient, BIC and IIR mechanisms, this implies that any other mechanism that is efficient, BIC, and interim IR will run an expected deficit. If there is no ex ante BB mechanism then there is no ex post BB mechanism either (deficit in expectation means that there must be deficit for at least some type realizations). Therefore, no feasible mechanism exists in this case.

3. Let's use the same mechanism. The budget surplus is zero conditional on no trade, and

$$[v_2 \wedge v_3] \vee c - [v_2 \vee v_3] \wedge \bar{c} \geq [v_2 \wedge v_3] - \bar{c}$$

   conditional on trade. So the expected budget surplus is bounded below by

$$\mathbb{E}([v_2 \wedge v_3] - \bar{c} | v_2 \vee v_3 \geq c) \geq \mathbb{E}([v_2 \wedge v_3] - \bar{c}) \geq 0.$$

   So the gVCG mechanism runs an expected budget surplus, and we can make the mechanism exactly ex ante budget balanced by distributing the expected revenue among players.

4. By analogous reasoning, the expected budget surplus is bounded below by

$$\mathbb{E}\left(v^{(2)} - \bar{c}\right),$$

   where $v^{(2)}$ is the second order statistic (the second-highest value). Since $v \in (\bar{c}, \bar{v})$ with positive probability under F, the expected value $\mathbb{E}v^{(2)}$ can be made arbitrarily close to $\bar{v}$ by taking N large enough; in particular, N can be chosen large enough to make it higher than $\bar{c}$.

[1] This problem is more complex due to two buyers instead of one, but the idea is the same. Further, while you cannot compute the expectation $\mathbb{E}_{\theta_{-i}}[\sum v_i(k^*(\theta), \theta_i)]$ in this problem – since distributions F and G are not specified, – you can still see that, e.g., when you try to compute B2's LCT $\tilde{v}_2 \in \arg\min_{v_2} \{\mathbb{E}_{c,v_3}[\max\{c, v_2, v_3\}]\}$ the expression inside the expectation is weakly decreasing in v_2 for all c, v_3, hence the whole expectation is also weakly decreasing, so it is minimized by $v_2 = 0$.