Exercises after Lecture 10 (M7): Matching models.

Problem 1: Solve your own problem

This problem is meant to demonstrate the power of DA algorithm, which finds a stable matching in *any* marriage market.

Consider a market with four men and four women. Come up with arbitrary preferences for all players (i.e., a ranking for each player of all players on the other side of the market and the option to stay single).

1. Find a stable matching generated by men-proposing DA algorithm.
2. Find a stable matching generated by women-proposing DA algorithm.
3. Are there any other stable matchings?
4. Suppose a men-proposing DA algorithm is run. Is there a profitable deviation for any of the women – i.e., can any woman misreport her preferences to the mechanism to improve her matching? If yes, show it; if not, explain why.

   (*Hint: such a deviation exists if and only if you have more than one stable matching, which happens if and only if the outcomes of W-DA and M-DA algorithms are different.*)

Problem 2: College admissions

This problem demonstrates how marriage model can be extended to allow many-to-one matchings, which turns it into a "college admissions model".

There is a market with four students $S = \{s_1, ..., s_4\}$ and three colleges $C = \{c_1, c_2, c_3\}$. College c_1 can admit two students (its *quota* is $q_1 = 2$); the remaining two colleges can admit one student each ($q_2 = q_3 = 1$). Players' preferences (ordinal rankings, written best to worst) are given by

$$\begin{aligned} \succ_{s_1}&: c_3, c_1, c_2 & \succ_{c_1}&: s_1, s_2, s_3, s_4 \\ \succ_{s_2}&: c_2, c_1, c_3 & \succ_{c_2}&: s_1, s_2, s_3, s_4 \\ \succ_{s_3}&: c_1, c_3, c_2 & \succ_{c_3}&: s_3, s_1, s_2, s_4 \\ \succ_{s_4}&: c_1, c_2, c_3 \end{aligned}$$

Your goal is to find a stable matching in this problem. The only difference from the marriage model we considered in class is that college c_1 can admit *two* students. The trick is to represent the two available spots in c_1 as two independent players which have the same preferences over students and which rank equally against other colleges among the students.

In particular, consider instead a market with the same four students but now four colleges $C' = \{c_{1.1}, c_{1.2}, c_2, c_3\}$ (each with quota $q_i = 1$, as in the marriage model), and preferences are given by

$$\begin{aligned} \succ_{s_1}&: c_3, c_{1.1}, c_{1.2}, c_2 & \succ_{c_{1.1}}&: s_1, s_2, s_3, s_4 \\ \succ_{s_2}&: c_2, c_{1.1}, c_{1.2}, c_3 & \succ_{c_{1.2}}&: s_1, s_2, s_3, s_4 \\ \succ_{s_3}&: c_{1.1}, c_{1.2}, c_3, c_2 & \succ_{c_2}&: s_1, s_2, s_3, s_4 \\ \succ_{s_4}&: c_{1.1}, c_{1.2}, c_2, c_3 & \succ_{c_3}&: s_3, s_1, s_2, s_4 \end{aligned}$$

1. Use the college-proposing DA algorithm to find a stable matching.
2. Matching μ generated by the C-DA algorithm is C'-optimal. However, there is another matching $\mu' = \{(c_1, s_2, s_4), (c_2, s_1), (c_3, s_3)\}$ that is strictly preferred to μ by all colleges in C. How can you explain this contradiction?

Problem 3: Book giveaway

Djul has defended his Ph.D. and found a job. He looks back at the small library of books that he has assembled during his studies and decides that he does not need them as much any more. Therefore, he decides to give the books away to fellow Ph.D. students. Suppose there are $b \in \{1, ..., B\}$ books and $i \in \{1, ..., N\}$ interested students. Since $N > B$, Djul decides that it would be fair to limit the giveaway to one book per person. Let $\theta_{i,b}$ denote the valuation of student i for book b (privately known by student i). Assume that all students are economists who act in pure self-interest.

1. Given that Ph.D. students are poor,[1] and Djul himself now has a well-paying job, he would prefer to give the books away for free. Propose a mechanism that Djul could use to allocate the books among fellow students for free and in a way that would be Pareto optimal.
2. Suppose now that $N = 6$, $B = 4$, and the realized valuations are as given in Table 1. Calculate the allocation produced by your mechanism from part 1.

$\theta_{i,b}$	$i = 1$	$i = 2$	$i = 3$	$i = 4$	$i = 5$	$i = 6$
$b = 1$	4	4	1	8	9	9
$b = 2$	0	2	4	9	5	3
$b = 3$	9	5	5	2	6	4
$b = 4$	7	6	0	7	2	6

Table 1: Preferences for the Book Giveaway problem

3. Does there exist a mechanism that allocates the books without transfers efficiently (i.e., in a welfare-maximizing way)? If yes: present a mechanism. If not: explain why.
4. Djul has run your mechanism from part 1 and messaged people regarding who got which book, but lost his phone with all the notes and messages before actually giving any books away. He thus cannot remember which book was promised to which student. Each student, however, knows which book they were promised. How can Djul recover the promised allocation without running the whole mechanism again? (Propose a mechanism that relies on students' reports of the books they were promised and explain why it works.)

[1] The story is taking place in the U.S.