# Exercises for Lecture 12: Information design.

## Problem 1: Informative Advertising as Persuasion

A consumer is choosing between two Samsung smartphones: the new Galaxy Fold, which costs $p_F = \$2000$, and the older Galaxy S10, which costs $p_S = \$1000$. The consumer does not know which of the two is right for her, and she is very afraid of making the wrong choice.

Formally, from the consumer's point of view, one of the two states is possible: $\omega \in \{F, S\}$. Her expected utility from buying phone $a \in \{F, S\}$ is given by

$$v_1(a|\phi) = \mathbb{E}_\omega \left[ w(a, \omega) \mid \phi \right] - p_a,$$

where $\phi$ denotes the probability that the consumer assigns to state being $\omega = F$, and the state-dependent valuations $w(a, \omega)$ are given by

| $w(a,\omega)$ | $\omega = F$ | $\omega = S$ |
|---|---|---|
| $a = F$ (buy Fold) | 3000 | 0 |
| $a = S$ (buy S10) | 0 | 1500 |

The consumer always has the option (denoted as $a = \varnothing$) to walk away from the purchase, which yields utility zero in both states.

The seller can procure the phones at zero cost, hence his profit $v_0(a)$ is given by

$$v_0(a) = \begin{cases} p_F & \text{if } a = F; \\ p_S & \text{if } a = S; \\ 0 & \text{if } a = \varnothing. \end{cases}$$

1. Describe the consumer's optimal choice rule $a(\phi)$ for any given belief $\phi = \mathbb{P}(\omega = F)$.
2. Write down the consumer's expected utility $V_1(\phi) = \max_a v_1(a|\phi)$ from following this optimal choice rule $a(\phi)$.
3. Write down the company's profit $V_0(\phi)$ from the consumer following her optimal choice rule $a(\phi)$.

Suppose that the consumer's prior is $\phi_0 = \frac{1}{2}$. The seller decides to engage in Bayesian Persuasion: he designs a quiz that, when passed by the consumer, will tell her which phone is likely better for her. Formally, a quiz is an experiment $\mu = \{(\tau_1, \phi_1), (\tau_2, \phi_2), ...\}$, which moves the consumer's belief to $\phi_k$ with probability $\tau_k$. Naturally, it must be that $\sum_k \tau_k = 1$ and $\sum_k \tau_k \phi_k = \phi_0$. Note that posteriors $\phi_k$ need not be in $\{0, 1\}$: the quiz may induce any posterior belief $\phi_k \in [0, 1]$.

4. Find the quiz/experiment $\mu$ that maximizes the seller's expected profit.
   *Hint: drawing a graph of $V_0(\phi)$ may help you.*

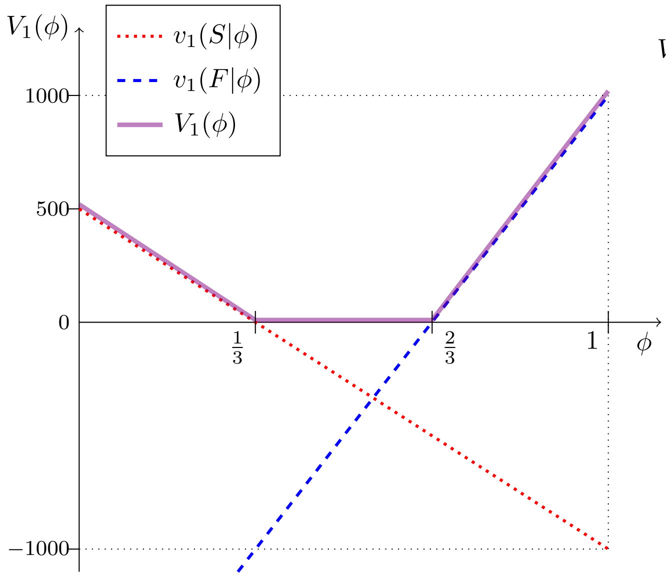


Figure 1: $V_1(\phi)$

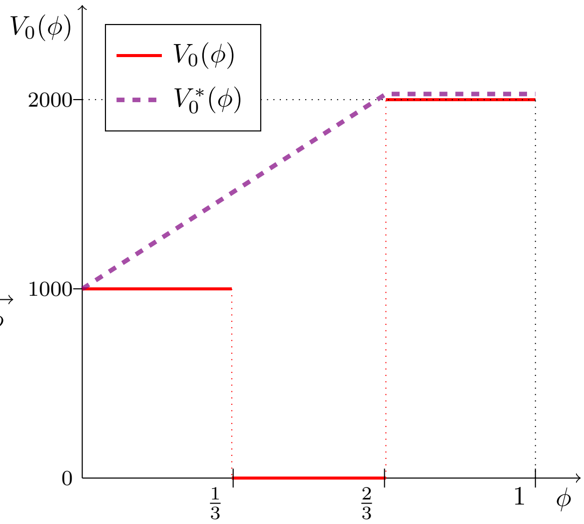


Figure 2: $V_0(\phi)$ and $V_0^*(\phi)$

## Solution

1. The consumer's utilities from the three options are given by:

$$\begin{aligned} v_1(F|\phi) &= \phi \cdot 3000 + (1-\phi)\cdot 0 - 2000 \\ v_1(S|\phi) &= \phi \cdot 0 + (1-\phi)\cdot 1500 - 1000 \\ v_1(\varnothing|\phi) &= 0 \end{aligned}$$

   The three are depicted in Figure 1. Taking the maximum of the three for a given $\phi$ yields the optimal choice rule

$$a(\phi) = \begin{cases} F & \text{if } \phi \geq \frac{2}{3}; \\ \varnothing & \text{if } \phi \in \left[\frac{1}{3}, \frac{2}{3}\right]; \\ S & \text{if } \phi \leq \frac{1}{3}. \end{cases}$$

2. From the previous answer, we get

$$V_1(\phi) = 1000 \cdot \begin{cases} 3\phi - 2 & \text{if } \phi \geq \frac{2}{3}; \\ 0 & \text{if } \phi \in \left[\frac{1}{3}, \frac{2}{3}\right]; \\ 0.5 - 1.5\phi & \text{if } \phi \leq \frac{1}{3}. \end{cases}$$

3. From (1), we have

$$V_0(\phi) = 1000 \cdot \begin{cases} 2 & \text{if } \phi \geq \frac{2}{3}; \\ 0 & \text{if } \phi \in \left[\frac{1}{3}, \frac{2}{3}\right]; \\ 1 & \text{if } \phi \leq \frac{1}{3}. \end{cases}$$

4. As suggested by the hint, look at the graph of $V_0(\phi)$ depicted in Figure 2. The profit $V_0^*(\phi)$ that the seller can achieve under the optimal Bayesian Persuasion mechanism is given by the smallest concave envelope of $V_0(\phi)$. You can see from the Figure that $V_0^*(\phi)$ coincides with $V_0(\phi)$ for $\phi \in NP \equiv \{0\} \cup [2/3, 1]$ – if the consumer's prior belonged to this set then no persuasion mechanism could increase

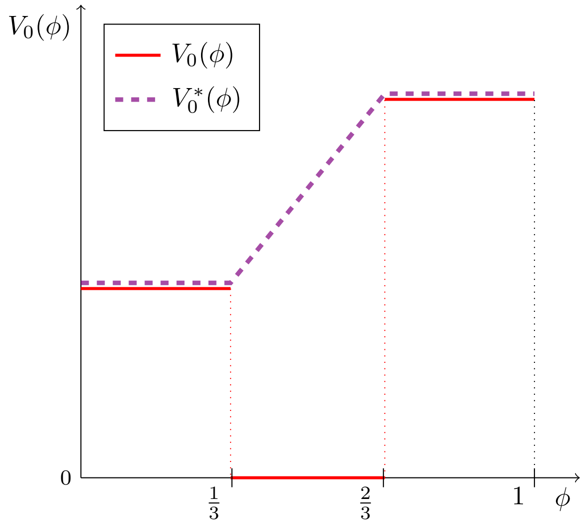


Figure 3: Incorrect $V_0^*(\phi)$

the seller's profit. For any remaining prior (which includes our case, $\phi_0 = 1/2$), the optimal persuasion mechanism splits the prior between two closest points in $NP$. In case of prior $\phi_0 = 1/2$, the optimal persuasion mechanism prescribes posteriors $\phi_1 = 0$ and $\phi_2 = 2/3$. The probabilities of these posteriors can then be computed from the consistency requirement (a.k.a. law of total probability):

$$\tau_1 \cdot \phi_1 + \tau_2 \cdot \phi_2 = \phi_0$$
$$\Leftrightarrow \tau_1 \cdot 0 + \tau_2 \cdot \frac{2}{3} = \frac{1}{2}$$

and the requirement $\tau_1 + \tau_2 = 1$. The two together yield $(\tau_1, \tau_2) = (1/4, 3/4)$. Hence the optimal experiment $\mu$ induces posterior $\phi_1 = 0$ with probability $\tau_1 = 1/4$ and posterior $\phi_2 = 2/3$ with probability $\tau_2 = 3/4$.

Note: graph in Figure 3 is **not** considered correct (since the resulting $V_0^*(\phi)$ is not concave).

## Problem 2: Two approaches to information design

Consider the following information design problem. There are two possible states, $\omega \in \{L, R\}$, the common prior belief that the state is $R$ is $\phi_0 = \mathbb{P}(\omega = R) = 1/2$. There is one player (receiver) and two actions $a \in \{u, d\}$ available to him. The receiver's payoffs as a function of state are given by the function $v_1(a, \omega)$, which is defined as

| $v_1(a,\omega)$ | $\omega = L$ | $\omega = R$ |
|---|---|---|
| $a = u$ | 3 | 0 |
| $a = d$ | 0 | 1 |

There is a designer who (before getting to observe $\omega$) designs an experiment that will send a message to the receiver, which may be informative about the true state $\omega$. The designer's payoff coincides with that of the receiver, with one exception: the designer receives a bribe of 4 if action $a = d$ is chosen in state $\omega = L$. In other words, the designer's payoff function $v_0(a, \omega)$ is given by

| $v_0(a,\omega)$ | $\omega = L$ | $\omega = R$ |
|---|---|---|
| $a = u$ | 3 | 0 |
| $a = d$ | 4 | 1 |

1. Derive the receiver's optimal action rule $\hat{a}(\phi)$, which maximizes his expected payoff, as a function of $\phi$, his posterior belief about the state after observing message $m$ generated by the experiment ($\phi = \mathbb{P}(\omega = R|m)$).
2. Derive and plot the designer's payoff function $V_0(\phi) \equiv \mathbb{E}_{\phi(\omega)}[v_0(\hat{a}(\phi), \omega)]$ as a function of the receiver's posterior $\phi$.
3. Derive and plot (on the same graph) the concave closure $V_0^*(\phi)$ of the designer's payoff function $V_0(\phi)$.
4. By looking at the plots of $V_0(\phi)$ and $V_0^*(\phi)$ and recalling that $\phi_0 = 1/2$, answer the following: what is the set of posteriors $\{\phi_1, \phi_2, ...\}$ induced by the optimal experiment (the one that maximizes the designer's expected payoff)? What is the designer's payoff from the optimal experiment?
5. Use the "correlated equilibria approach" to find the optimal experiment. In particular, find a decision rule $\sigma : \{L, R\} \to \Delta(\{u, d\})$ (so $\sigma(u|\omega)+\sigma(d|\omega) = 1$ for any $\omega$) which maximizes the designer's expected payoff as given by

$$v_0^*(\sigma) \equiv \sum_{a,\omega} v_0(a, \omega)\sigma(a|\omega)\phi(\omega)$$

subject to the obedience constraint: for any $a, a' \in \{u, d\}$,

$$\sum_{\omega} v_1(a, \omega)\sigma(a|\omega)\phi(\omega) \geq \sum_{\omega} v_1(a', \omega)\sigma(a|\omega)\phi(\omega).$$

## Solution

1. The receiver's expected utility of selecting $a = u$ is $\mathbb{E}_\phi v_1(u, \omega) = 3(1 - \phi) + 0\phi$, while for $a = d$ it is $\mathbb{E}_\phi v_1(d, \omega) = 0(1 - \phi) + 1\phi$. Taking the maximum of the two, the optimal action is

$$\hat{a}(\phi) = \begin{cases} u & \text{if } \phi < 3/4; \\ d & \text{if } \phi \geq 3/4. \end{cases}$$

Figure 4 plots utilities from both actions and the optimal action.

2. Plugging the optimal action $\hat{a}(\omega)$ into the designer's payoff function and taking expectations w.r.t. $\phi(\omega)$ (the *receiver's* posterior), we get

$$V_0(\phi) = \begin{cases} 3 - 3\phi & \text{if } \phi < 3/4; \\ 4 - 3\phi & \text{if } \phi \geq 3/4. \end{cases}$$

Note that we did indeed break the receiver's indifference in the designer's favor. This function is plotted in Figure 5.

3. See Figure 5.
4. The set of optimal posteriors is given the set of $\phi$ that are such that $V_0^*(\phi) = V_0(\phi)$. Starting from $\phi_0 = 1/2$, we see from Figure 5 that the two closest such posteriors are $\phi = 0$ and $\phi = 3/4$.

Designer's optimal payoff is given by

$$V_0^*(\phi) = \begin{cases} 3 - \frac{5}{3}\phi & \text{if } \phi < 3/4; \\ 4 - 3\phi & \text{if } \phi \geq 3/4. \end{cases}$$

(since it is a piecewise-linear function passing through points $(\phi, V_0^*) = \{(0, 3); (3/4, 7/4); (1, 1)\}$). Plugging in the prior $\phi_0$, we get that the designer's expected payoff given this prior is $3 - 5/6 = 13/6$.

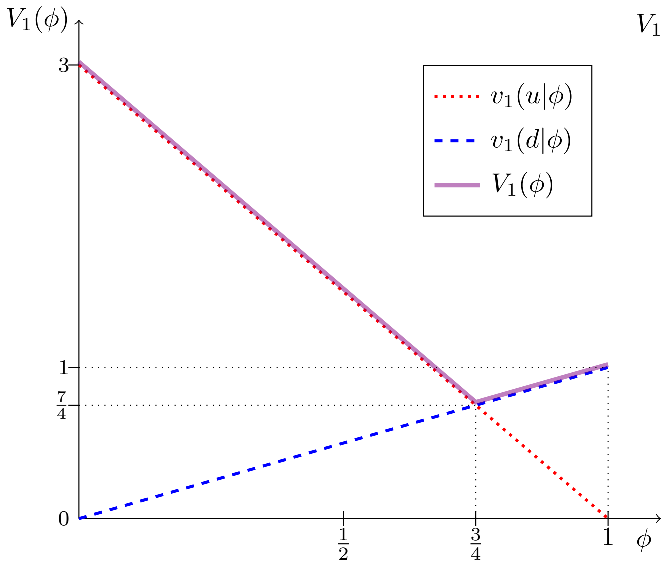


Figure 4: $V_1(\phi)$

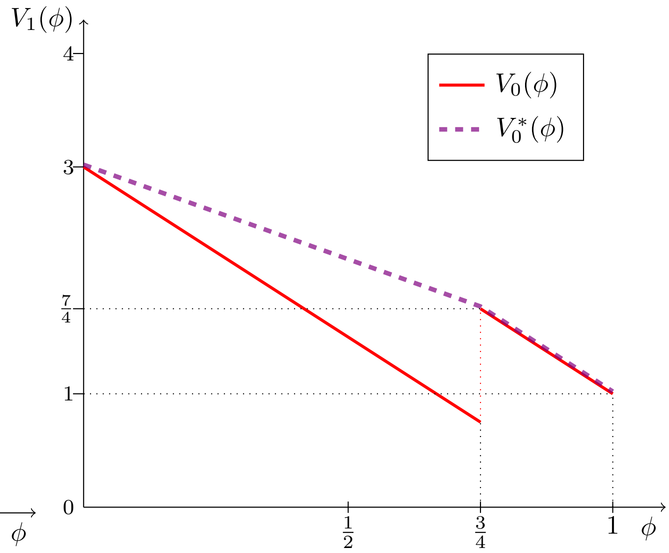


Figure 5: $V_0(\phi)$ and $V_0^*(\phi)$

5. The designer's problem is:

$$\begin{aligned}
&\max_{\sigma}\left\{(3\sigma(u|L)+4\sigma(d|L))\frac{1}{2}+(0\sigma(u|R)+1\sigma(d|R))\frac{1}{2}\right\}\\
&\text{s.t. } 3\sigma(u|L)\frac{1}{2}+0\sigma(u|R)\frac{1}{2}\geq 0\sigma(u|L)\frac{1}{2}+1\sigma(u|R)\frac{1}{2}\\
&\quad 0\sigma(d|L)\frac{1}{2}+1\sigma(d|R)\frac{1}{2}\geq 3\sigma(d|L)\frac{1}{2}+0\sigma(d|R)\frac{1}{2}\\
&\quad \sigma(u|L)+\sigma(d|L)=1\\
&\quad \sigma(u|R)+\sigma(d|R)=1
\end{aligned}$$

(plus the implicit constraints $\sigma(a|\omega)\in[0,1]$ for all $a,\omega$). After scaling the objective function and the first two constraints up by a factor of 2 (to get rid of irrelevant $\frac{1}{2}$s), omitting the zeroes, and then also expressing $\sigma(d|\omega)=1-\sigma(u|\omega)$ for $\omega\in\{L,R\}$ from the two last constraints, the problem reduces to:

$$\begin{aligned}
&\max_{\sigma(u|L),\sigma(u|R)}\{3\sigma(u|L)+4(1-\sigma(u|L))+(1-\sigma(u|R))\}\\
&\text{s.t. } 3\sigma(u|L)\geq\sigma(u|R)\\
&\quad 1-\sigma(u|R)\geq 3(1-\sigma(u|L))
\end{aligned}$$

or, equivalently,

$$\begin{aligned}
&\max_{\sigma(u|L),\sigma(u|R)}\{5-\sigma(u|L)-\sigma(d|R)\}\\
&\text{s.t. } \sigma(u|R)\leq 3\sigma(u|L)\\
&\quad \sigma(u|R)\leq 3\sigma(u|L)-2
\end{aligned}$$

So we want to select $\sigma(u|R)$ and $\sigma(u|L)$ as low as possible. It is immediate that $\sigma(u|R)=0$ is optimal. The first obedience constraint is then satisfied automatically (again, given the implied constraint $\sigma(u|L)\geq 0$). From the second constraint we get $\sigma(u|L)\geq 2/3$, thus in the optimum $\sigma(u|L)=2/3$.

In the end, the solution is:

$$\sigma(u|L) = 2/3 \qquad \sigma(d|L) = 1 - \sigma(u|L) = 1/3;$$
$$\sigma(u|R) = 0 \qquad \sigma(d|R) = 1 - \sigma(u|R) = 1.$$

## Problem 3: Sequential persuasion and Proposition 22

**Background:** A debate has been ongoing (at least in the US) for the past few years on whether the gig economy workers (Uber drivers, Wolt couriers, etc) can be classified as independent contractors, as they currently are, or must be enlisted as proper employees. The latter would mean that the company would have to provide such workers with minimal wage, health insurance, paid vacations and other social benefits.[1]

Voters in California have, in November 2020, "overwhelmingly approved" the so-called Proposition 22, which would allow the gig economy firms to continue classifying its workers as contractors (P22 is an amendment to an earlier legislation that would have required a reclassification of workers). Companies' opponents are disappointed with the outcome, blaming it partially on the fact that the companies managed to spend 10 times more money on advertising and promoting their viewpoint.[2]

**Problem:** Consider a setting with three players: a representative voter, a firm, and a worker union. Suppose a vote on Proposition 22 is coming. The true state $\omega \in \{0, 1\}$ represents whether adopting this regulation is socially beneficial. The voter does not know $\omega$ but wants to choose the right thing: $a \in \{0, 1\}$, $v_v(a, \omega) = \mathbb{I}\{a = \omega\}$. The two other parties want to tilt this decision in their favor: the firm's utility function is $v_f(a, \omega) = \mathbb{I}\{a = 1\}$, while the workers union's utility function is $v_u(a, \omega) = \mathbb{I}\{a = 0\}$. (As usual, $\mathbb{I}(\cdot)$ is the indicator function.)

To affect the voter's decision, the firm and the union engage in Bayesian Persuasion, i.e., they can each select any distribution of messages $\mu(m|\omega)$.[3] The firm's budgetary advantage means it moves after the union and can say the final word. There are thus three stages in the problem:

(i) the union selects a state-contingent distribution of messages $\mu_u(m_u|\omega)$; then a message $m_u$ is drawn from this distribution and is observed by all parties;

(ii) the firm selects a state-contingent distribution of messages $\mu_f(m_f|\omega)$; then a message $m_f$ is drawn from this distribution and is observed by all parties;

(iii) the voter selects an action $a$.

We will solve this problem by backwards induction. Answer the following questions. *Hint: drawing graphs of every object you calculate can be helpful in this problem.*

1. Let $\phi_2 \equiv \mathbb{P}(\omega = 1|s_u, s_f)$ denote the probability that the voter's posterior belief assigns to state $\omega = 1$ after observing both messages $m_u, m_f$. Derive the optimal action rule $\hat{a}(\phi_2) \equiv \arg\max_a \mathbb{E}_\omega[v_v(a, \omega)|\phi_2]$ which maximizes the voter's expected utility, as a function of $\phi_2$.

2. Calculate the expected utility $V_f(\phi_2) \equiv \mathbb{E}_\omega[v_f(\hat{a}(\phi_2), \omega)]|\phi_2]$ that the firm receives from the voter's optimal choice conditional on voter's posterior belief $\phi_2$.

3. Let $\phi_1 \equiv \mathbb{P}(\omega = 1|s_u)$ denote the probability that the voter's belief assigns to state $\omega = 1$ after observing mesasge $m_u$. The firm's problem of selecting an optimal communication strategy $\mu_f(m_f|\omega)$

[1] You can find some broad overview of the issue here: https://arstechnica.com/tech-policy/2019/09/uber-and-lyft-vow-continued-fight-against-california-worker-rights-bill/.

[2] https://arstechnica.com/tech-policy/2020/11/uber-and-lyft-in-driving-seat-to-remake-us-labor-laws/

[3] You can interpret Bayesian Persuasion in many ways in this setting. One way is generating media attention: the firm and the union can make the voter pay attention to the issue think about it, and can steer the voter's belief about the state to some extent, but they cannot directly control what conclusions the voter arrives to. Another interpretation is that the firm and the union commission research (academic or journalistic), but have no direct control over its conclusions.

is equivalent to choosing a distribution of posteriors $Q_f(\phi_2|\phi_1)$. Derive $Q_f$ that maximizes the firm's expected profit.

4. Calculate the expected utility $V_u(\phi_1) \equiv \mathbb{E}_\omega[v_u(\hat{a}(\phi_2), \omega)]|\phi_1]$ that the union receives from the voter's optimal choice conditional on voter's belief $\phi_1$.

5. Let $\phi_0 \equiv \mathbb{P}(\omega = 1)$ denote the probability that the voter's prior belief assigns to state $\omega = 1$.The union's problem of selecting an optimal communication strategy $\mu_u(m_u|\omega)$ is equivalent to choosing a distribution of posteriors $Q_u(\phi_1|\phi_0)$. Derive $Q_u$ that maximizes the union's expected profit.

6. What can you say about the informational outcome for the voter? (I.e., what information does the voter have in the end?) Would it be different if the two senders moved in the opposite order or simultaneously? (Make a convincing intuitive argument.)

7. We are interested in evaluating the union's complaint, which goes as follows:

   ``These corporations spent over $200 million on a corporate misinformation, deceptive campaign to rig our democratic process and to continue their exploitation of working people. It is a blasphemy and a sin.''

   Would you say that in the information design problem that you have solved, the firm's communication interfered with the voter's decision process? Would you say that this problem captures accurately the essence of the complaint (i.e., the effect of larger campaign expenditure)? If not, how would you set up a model that captures it better?

## Solution

1. $\mathbb{E}_\omega[v_v(a, \omega)|\phi_2] = \mathbb{P}(\omega = a)$, hence choosing $a = 1$ yields expected utility $\phi_2$, and choosing $a = 0$ yields $1 - \phi_2$. The optimal action is then $\hat{a}(\phi_2) = \mathbb{I}(\phi_2 \geq 0.5)$.

2. The firm's utility function is $v_f(a, \omega) = \mathbb{I}\{a = 1\}$, hence the expected utility from the voter's decision is $V_f(\phi_2) = \mathbb{I}\{\phi_2 \geq 0.5\}$.

3. The concave closure of $V_f(\phi_2)$ is $V_f^*(\phi_2) = \min\{2\phi_2, 1\}$ (draw a graph to see this). Therefore, $V_f$ and $V_f^*$ coincide on $\phi_2 \in \{0\} \cup [0.5, 1]$ – if $\phi_1$ belongs to this set, an uninformative experiment is optimal. If instead $\phi_1 \in (0, 0.5)$ then it is split into posteriors $\phi_2 \in \{0, 0.5\}$. We can use the law of total probability to find the optimal experiment for that case, eventually yielding the following distributions $Q_f(\phi_2|\phi_1)$:

$$\phi_2 = \begin{cases} \phi_1 \text{ w.p. } 1, & \text{if } \phi_1 \in \{0\} \cup [0.5, 1]; \\ \begin{cases} 0.5 & \text{w.p. } 2\phi_1, \\ 0 & \text{w.p. } 1 - 2\phi_1 \end{cases} & \text{if } \phi_1 \in (0.5, 1). \end{cases}$$

4. Given the firm's strategy above, the union expects that the voter will choose $a = 1$ w.p. $\min\{2\phi_1, 1\}$ and $a = 0$ otherwise. Therefore, $V_u(\phi_1) = 1 - \min\{2\phi_1, 1\} = \max\{1 - 2\phi, 0\}$.

5. Applying the same process as above: the concave closure of $V_u$ is $V_u^*(\phi_1) = 1 - \phi_1$. So $V_u$ and $V_u^*$ only coincide on $\phi_1 \in \{0, 1\}$, hence the union's optimal experiment $Q_u(\phi_1|\phi_0)$ will be perfectly informative:

$$\phi_1 = \begin{cases} 1 & \text{w.p. } \phi_0, \\ 0 & \text{w.p. } 1 - \phi_0. \end{cases}$$

6. The voter perfectly learns the state from the union's message. Note that the only reason the union provides perfect information to the voter is the implicit threat of the firm then providing any missing information if the union tries to conceal any information unfavorable to the union's cause. This outcome

stems from the firm's and the union's incentives being the complete opposites, and would persist so long as both of them are able to design their experiments, regardless of whether one signal is sent after the other or both are selecting their experiments simultaneously.[4]

7. The argument above implies that it does not matter which of the two parties have the last-mover advantage, hence in our model the funding advantage is irrelevant for the outcome. However, one may easily argue that our model does not fully capture the funding advantage, and it affects other aspects as well. For example, it could be the case instead (and would be more plausible) that the funding affects the set of experiments available to the sender. I.e., the firm having spent more on advertising would mean that the firm can select more informative signals than the union – in which case it could indeed be the case that the voter in equilibrium observes more information favorable for the firm than for the union.

[4] A general treatment of the problem with multiple senders and simultaneous moves is available in Gentzkow and Kamenica (REStud, 2016).