Financial Markets Microstructure

Lecture 10

Limit order book, part 2
Chapter 6.2-6.3 of FPR

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Last time

- Glosten model: see how the behavior of competitive liquidity providers in a LOB is different from dealers' behavior.
- Pricing rule: marginal price of qth unit (on the ask side) is

$$p(q) = \mathbb{E}[v|x \geq q].$$

- Reminder: in a dealer market, average price when q units are traded is

$$\bar{p}(q) = \mathbb{E}[v|x = q].$$

This lecture:

1. Static Analysis: Glosten Model (discrete ticks)

2. Market design

Ticks

- The version of the Glosten model we've seen last time outlines some basic differences between LOB and dealer markets
  - Adverse selection affects prices differently
- It neglects the discreteness of prices:
  - Often prices are discrete and must lie at a tick – tick size is the increment b/w prices
  - E.g. at NYSE it was $1/8 for stocks with prices over one dollar until June 1997, when, under regulatory pressure, it was reduced to $1/16 and finally, in 2000, to one cent.
- Q for today: how does this discreteness (and tick size in particular) affect market outcomes?
  - Prioritized limit orders become profitable when there are ticks (since no 'marginal undercutting')

Discrete Glosten model: Setup

- **Asset**: Continue with single asset with value $v \sim G$
- **Market order** x correlated with v (reminder: notation different from the book)
  - Unconditional c.d.f. $F(x)$
  - Again, focus on the ask side of the book, $x > 0$
- **Discrete price grid**
  - A_1 is lowest price tick above μ
  - $A_k - A_{k-1} > 0$ is the tick size
- **Limit orders**
  - *Time priority*: first posted, first executed
  - *Display cost*: C per unit (paid regardless of whether order executes)
  - Let q_k denote the *cumulative* volume supplied (depth) at prices up to A_k

Discrete model: Equilibrium

- **Competition**: Limit orders are supplied at each tick until the last order earns zero profit
- **Zero-profit condition**:

$$\mathbb{P}(x \geq q_k) \cdot \left[A_k - \mathbb{E}[v|x \geq q_k]\right] - C = 0,$$

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solved by

$$A_k = \underbrace{\mathbb{E}[v|x \geq q_k]}_{\text{Adverse selection}} + \underbrace{\frac{C}{\mathbb{P}(x \geq q_k)}}_{\text{Execution risk}}$$

(Though we actually want to solve for endogenous depth q_k given exogenous price ticks A_k)

Discrete Glosten model: comments

- The pricing rule seems to be exactly the same as in the continuous model...
- ...but this is only for the marginal units!

Either way, let's now look at a few examples to practice applying this pricing rule!

Example 1: Setup

- **Asset**. Let g be the marginal distribution of G and

$$g(v) = \begin{cases} 1/2 \text{ if } v = v^H; \\ 1/2 \text{ if } v = v^L, \end{cases}$$

  with $v^H = \mu + \sigma$ and $v^L = \mu - \sigma$.
- **Traders**. Single trader, who uses a market order.
  - Prob. π: risk-neutral speculator (S) who knows v
  - Prob $1 - \pi$: noise trader (N) who buys/sells with equal probability, and uses large (x_L) or small ($x_S < x_L$) order with equal probability:

$$\mathbb{P}(x = x_S|N) = \mathbb{P}(x = x_L|N) = \mathbb{P}(x = -x_S|N) = \mathbb{P}(x = -x_L|N) = 1/4$$

- **No display cost**. Let $C = 0$
- **Continuous prices**.

Example 1: Equilibrium

- **Equilibrium**: Look for eq. with $q_1 = x_S$ and $q_2 = x_L$ for some ask prices $\mu < A_1 < A_2 < v^H$. In this prb: q_1, q_2 given, we look for A_1, A_2.

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- **Speculator**:
  - If $x > x_L$, speculator reveals himself – never optimal
  - Since $v^H > A_1, A_2 > \mu$, if $v = v^H$ then speculator buys x_L units; (and if $v = v^L$ then sells x_L).
    Why is it not optimal to shade the order?

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  - Shading order (strategically restricting trade size) is not optimal for the speculator because buying more does not worsen the price of the previous units, unlike in dealer mkt
- **Price**. In equilibrium, price must equal $\mathbb{E}[v|x \geq q_k]$:

$$A_1 = \mathbb{E}[v|x \geq x_S] = \mu + \pi\sigma,$$
$$A_2 = \mathbb{E}[v|x \geq x_L] = \mu + \frac{2\pi}{1+\pi}\sigma$$

- Obvious that indeed, $\mu < A_1 < A_2 < v^H$. Thus: equilibrium.

Example 1: Comment

- Not the best example for discreteness:
  - The example does *not* assume fixed ticks...
  - But they arise endogenously in equilibrium...
  - Due to discreteness of noise traders' strategy
  - (A very artificial assumption)

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- Not the best example for discreteness:
  - The example does *not* assume fixed ticks...
  - But they arise endogenously in equilibrium...
  - Due to discreteness of noise traders' strategy
  - (A very artificial assumption)
- But focus on adverse selection leading to limited depth:
  - Price increases in order size
  - Not due to informed trader having stronger info, as in Kyle model
  - But due to noise traders' order becoming (relatively) less likely

Example 2: Setup

- **Market orders**: Exponential distribution, $f(x) = \frac{\theta}{2} e^{-\theta|x|}$
- **Asset**. Assume 'price impact' equation $\mathbb{E}[v|x] = \mu + \lambda x$, where $\lambda > 0$ is a constant measuring informativeness of order flow
  - Thus, we are taking a short-cut and modeling adverse selection in a 'reduced form': rather than modeling the informed traders, we model their price impact
- There is some order submission cost C
- **Goal**: find the equation connecting A_k and q_k (given arbitrary tick grid $A_1, \ldots, A_k, \ldots$)

Example 2: Equilibrium

- Focus on **ask side**: $q_k > 0$. For $x \geq q_k$:

$$
\begin{aligned}
f(x|x \geq q_k) &= \frac{f(x)}{\mathbb{P}(x \geq q_k)} \\
&= \frac{f(x)}{\int_{q_k}^{\infty} f(x)dx} \\
&= \frac{\frac{\theta}{2} \cdot e^{-\theta x}}{e^{-\theta q_k}/2} \qquad (|x| = x \text{ since } x \geq q_k > 0) \\
&= \theta \cdot e^{-\theta(x-q_k)} \\
&= e^{\theta q_k} \left[\theta \cdot e^{-\theta x}\right]
\end{aligned}
$$

Example 2: Equilibrium (2)

- The expected value at tick k becomes

$$
\begin{aligned}
\mathbb{E}[v|x \geq q_k] &= \mu + \lambda \mathbb{E}[x|x \geq q_k] \\
&= \mu + \lambda \int_{q_k}^{\infty} x \cdot f(x|x \geq q_k) dx \\
&= \mu + \lambda e^{\theta q_k} \int_{q_k}^{\infty} x \cdot \theta \cdot e^{-\theta x} dx \\
&= \mu + \lambda e^{\theta q_k} \left\{ \left[-x \cdot e^{-\theta x} \right]_{q_k}^{\infty} - \int_{q_k}^{\infty} -e^{-\theta x} dx \right\} \text{ (int. by parts)} \\
&= \mu + \lambda e^{\theta q_k} \left\{ \left[0 + q_k \cdot e^{-\theta q_k} \right] - \frac{1}{\theta} [0 - e^{-\theta q_k}] \right\} \\
&= \mu + \lambda \left(\frac{1}{\theta} + q_k \right)
\end{aligned}
$$

Example 2: Equilibrium (3)

- Hence, the ask price at tick k can be found by solving

$$\underbrace{\mathbb{P}(x \geq q_k)}_{e^{-\theta q_k}/2}[A_k - \underbrace{\mathbb{E}[v|x \geq q_k]}_{\mu+\lambda\left(\frac{1}{\theta}+q_k\right)}] - C = 0,$$

which gives

$$A_k = \mu + \lambda\left(\frac{1}{\theta} + q_k\right) + \frac{2C}{e^{-\theta q_k}}.$$

- (Again: we actually need the opposite – find q_k for a given tick A_k – but it is hard to get a closed-form expression for that.)

Glosten: Empirical evidence

- **Sandås [2001]** estimates Glosten model (in a form similar to example 2 above) via GMM, using intraday snapshots of LOB from Stockholm Stock Exchange and data on market orders
- Estimates the info content of market orders vs actual pricing schedules, so effectively the $\mathbb{E}[v|x \geq q]$ inferred from pricing schedule and the actual $\mathbb{E}[v|x = q]$ from the price dynamics.
- Zero profit condition is tested and rejected: LOB not deep enough to drive average expected profits to zero
- Also, estimated order execution costs are negative for the best bid and ask – i.e., those limit traders have some intrinsic preference for trading (although these days many exchanges do offer negative execution fees to limit traders to incentivize liq-ty provision)

Glosten model: Conclusion

- Limit traders act in the same capacity as the dealer did before
  - but face different informational environment
  - so act differently
  - which leads to different market outcomes
- With discrete ticks and time priority, even competitive limit traders can get positive expected profits

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Market design

- There are many dimensions in which legislation or exhange rules can regulate trade
- Today's phrase of the day: "unintended consequences"
  - Attempts to mitigate a particular inefficiency may have far-fetching consequences
  - We will look at a few examples

Tick size

- Assume time priority is the second order after price priority
  - I.e., first limit order posted at tick executes first
- Profit of the limit trader at price A_k is:
  - Zero for the marginal (last) limit order at A_k
  - Strictly positive for inframarginal orders because (1) order executes with higher probability and (2) info content of mkt order is weaker
- Q: what happens if we change the tick size?

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- Q: what happens if we change the tick size?
- This profit is reduced with smaller tick sizes
  - Hence in the long(er) run, decreasing tick size drives away limit traders and reduces depth
  - But it will also reduce spread (by design) and reduce trading costs for the opposite side of the market (liquidity takers)

Tick size

- Driving away limit traders intuitively also has dynamic repercussions
  - LOB is replenished more slowly after trades – so market orders traded more frequently against non-competitive prices

Tick size

- Driving away limit traders intuitively also has dynamic repercussions
  - LOB is replenished more slowly after trades – so market orders traded more frequently against non-competitive prices
- Goldstein and Kavajecz [2000] explored the NYSE 1997 case (tick size from \$1/8 to \$1/16)
  - Trading costs decreased for small orders
  - Unclear for large orders
  - Aligns with our predictions (smaller spread, smaller depth)

Priority rules

- With pro-rata allocation (limit orders at given tick executed proportionally to their size), as opposed to time priority:
  - The expected profit of all orders at price A_k must be zero (as opposed to strictly positive)
  - So execution probabilities must be lower for all orders
- Lower profits lead to the same consequences as with reducing tick size
  - Less liquidity provision in the long run
  - Lower LOB resiliency (slower replenishment)
- Pro-rata allocation rule used in the electronic futures markets for the leading short-term interest rate and for the two-year U.S. Treasuries.

Hybrid market

Hybrid market

- Suppose a dealer can compete with the limit order book, as follows
- The dealer may observe trade size x before serving the order (and can fulfill the order before it is matched against LOB)
- Can profit by pricing at $\mathbb{E}[v|x = q]$ rather than an average of $\mathbb{E}[v|x \geq y]$ for $y \leq q$ which is used by the LOB
- Especially profitable on small trades
- But the existence of such a dealer invalidates our analysis of the LOB
  - Profitable limit orders are being picked off
  - So limit traders would gain negative profits if they follow the old strategy $\Rightarrow$ incentive to change their strategy (or quit the market)
  - Liquidity demanders might also change their trading behavior

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- Example 1 from before continued. Assume an uninformed dealer receives order x. Can either send order to LOB or execute himself (at better price)

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- Focus on ask side. Let A_k^H be the hybrid ask price. When dealer observes $x = x_S$, he knows trader is noise trader and thus $\mathbb{E}[v|N] = \mu$.
  - Can execute order at just below A_1^H and earn profit $A_1^H - \mu$.

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- Focus on ask side. Let A_k^H be the hybrid ask price. When dealer observes $x = x_S$, he knows trader is noise trader and thus $\mathbb{E}[v|N] = \mu$.
  - Can execute order at just below A_1^H and earn profit $A_1^H - \mu$.
- Hence, only large orders $x = x_L$ are sent to LOB. LOB traders will expect this, and will price as if any order arriving to the LOB is large:

$$\mathbb{E}[v|x \geq x_S] = \mathbb{E}[v|x \geq x_L] = \mu + \frac{2\pi}{1+\pi}\sigma,$$

  and thus $A_1^H = A_2^H = \mu + \frac{2\pi}{1+\pi}\sigma$.
- $A_1^H > A_1$ and $A_2^H = A_2$: hybrid market less liquid than normal market

Hybrid market: conclusions 1

- To be fair: adding a dealer to LOB market...
  - decreases liquidity in "good times", when there would've been a thick LOB
  - but can help in bad times: if LOB is empty then adding a dealer has no adverse effects and will actually increase liquidity
  - So in the end, adding a dealer is like a liquidity insurance for the market
- Banks and other systemic internalizers with their dark pools can act as dealers too, picking off the good orders and forwarding the rest to the market.
  - Recent regulation heavily restricted banks and platforms in how much they can internalize their clients' trades.
- More analysis of hybrid markets with risk-averse traders: see Viswanathan and Wang [2002]

Hybrid market: conclusions 2

Analysis of the example above relied on a bunch of implicit assumptions (which are not necessarily true):

- Assumed the dealer had time priority over (could undercut all of) the LOB. If MO-traders can trade against the LOB before the dealer can act, the conclusions are different.
- Assumed the dealer is a monopolist – competitive dealers would yield different predictions.
- Assumed the market order revealed enough information. If MO-traders split their orders (trade one unit at a time), dealers no longer have any advantage. (Back and Baruch [2007])

Market Design: conclusion

- Regulation aimed at improving market liquidity can backfire by distorting agents' incentives

Next week

- Dynamic LOB analysis: traders can choose between limit and market orders

Homework

- Thinking in the framework of the discrete model: suppose tick size is actually zero; quotes can be placed in a continuous price space. Suppose that there is price priority. What then is the role of time priority, so that first-come quotes at identical prices are served first?
- Solve exercise 1 after ch.6 (pages 232-233) in the textbook. Note that you need to use Bayes' rule to assess the conditional distribution over v given a market order of size x (and work through slightly different notation)

References I

K. Back and S. Baruch. Working orders in limit order markets and floor exchanges. *The Journal of Finance*, 62(4):1589–1621, 2007. URL `https://doi.org/10.1111/j.1540-6261.2007.01252.x`. Publisher: Wiley Online Library.

M. A. Goldstein and K. A. Kavajecz. Eighths, sixteenths, and market depth: changes in tick size and liquidity provision on the NYSE. *Journal of Financial Economics*, 56(1):125–149, Apr. 2000. ISSN 0304-405X. URL `https://doi.org/10.1016/S0304-405X(99)00061-6`.

P. Sandås. Adverse Selection and Competitive Market Making: Empirical Evidence from a Limit Order Market. *The Review of Financial Studies*, 14(3):705–734, 2001. ISSN 0893-9454. URL `https://doi.org/10.1093/rfs/14.3.705`. Publisher: [Oxford University Press, Society for Financial Studies].

References II

S. Viswanathan and J. J. Wang. Market architecture: limit-order books versus dealership markets. *Journal of Financial Markets*, 5(2):127–167, 2002. Publisher: Elsevier.