Financial Markets Microstructure

Lecture 11

Limit order book (part 3)
Chapter 6.3-6.4 of FPR

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Last time

- First look into LOB markets using **?** (continuous and discrete versions).
- Limit traders act in the same capacity as the dealer did before
  - but face different incentives
  - so act differently
  - which leads to different market outcomes.
- First dive into market design: the effects of tick sizes, priority rules, and dealer interventions

Today

- Dynamic analysis of LOB markets:
  - How do traders choose between limit and market orders?

Tick size and time priority

Suppose that there is no tick; quotes can be placed in a continuous price space. Suppose that there is price priority. What is then the role of time priority, so that first-come quotes at identical prices are served first?

Dynamic analysis

- **?** shows how price schedule is formed under adverse selection
  - But this analysis is static – traders have no choice between limit/market orders = making/taking liquidity
- Proper dynamic analysis is quite difficult
- Choice between MO/LO depends on:
  1. Price differential (=bid-ask spread)
  2. Execution probability of a limit order
     - ...which depends on the current state of LOB
     - But also on next trader's choice between MO/LO
     - ...which depends on execution probability ...
  3. Adverse selection faced by limit orders
     - ...which depends on *who* will trade against my limit order
     - ...which depends on next trader's choice between market/limit order depending on their pvt info ...

Literature

- We will look at a very simple version of the **?** model
  - fixed ask/bid prices, endogenous choice between MO and LO
  - The book for some reason attributes it to **?**, which is a quite different model of LOB markets. Old class materials may also call this "the Parlour model"
- Other models:
  - **?** and **?**: models with heterogeneous discount factors; patient traders choose LO, impatient choose MO
  - **?**: combine all of the above in a single tractable model
  - Worth noting: limit orders are effectively free call/put options, so could be priced as such (**?**)

Foucault Model: setup

- **Exogenous prices**. Exogenous bid and ask prices $A > v > B$
- **Strategic traders**. One trader arrives per period, chooses b/w limit or market order (one unit)
  - LO valid for one period, then automatically cancelled. Choice depends on prob. of LO being executed, i.e. 'hit' by a MO from the next trader
  - Valuation: $v + y$, where $y \sim U[-Y, Y]$ is idiosyncratic valuation, private and independent across traders. v is known and common to all. Note: book assumes a binary distr-n $y \in \{-Y, Y\}$.
- **Noise traders**. Numerous and patient. Always submit limit orders at A (LSell) and/or B (LBuy). Prioritised *after* the strategic trader.
- **Goal**: derive the traders' optimal strategy

Profits

- Let $P_M^B (P_M^S)$ be the (endogenous) probability that the next trader submits a market buy (sell) order

$$\begin{aligned}
\text{Market sell: } & B - (v + y) \\
\text{Limit sell: } & (A - v - y) P_M^B \\
\text{Limit buy: } & (v + y - B) P_M^S \\
\text{Market buy: } & v + y - A
\end{aligned}$$

- Limit order more attractive when more likely to execute: IRL (not in this model), this leads to automatic tendency for the limit order book to be replenished (resiliency)

Analysis

- Look for stationary eqm where A, B, P_M^B, P_M^S are constant
- From the trader profits, we see that there must exist $\overline{y}$, $\underline{y}$ and $\hat{y}$ such that the best response (optimal order) of the trader is

$$BR = \begin{cases} MSell & \text{if} \quad -Y \leq y \leq \underline{y} \\ LSell & \text{if} \quad \underline{y} \leq y \leq \hat{y} \\ LBuy & \text{if} \quad \hat{y} \leq y \leq \overline{y} \\ MBuy & \text{if} \quad \overline{y} \leq y \leq Y \end{cases}$$

Traders with greater urgency/need to trade (extreme y) use market orders; "patient" traders use limit orders.

Equilibrium

- From trader's BR:

$$P_M^S = \mathbb{P}(-Y \leq y \leq \underline{y}) = \frac{\underline{y} + Y}{2Y}$$
$$P_M^B = \mathbb{P}(\overline{y} \leq y \leq Y) = \frac{Y - \overline{y}}{2Y}$$

- At $\underline{y}$, indifferent btw $MSell$ and $LSell$. At $\hat{y}$, indifferent btw $LSell$ and $LBuy$. At $\overline{y}$, indifferent btw $LBuy$ and $MBuy$. Thus:

$$B - (v + \underline{y}) = (A - v - \underline{y})\frac{Y - \overline{y}}{2Y} \tag{1}$$
$$(A - v - \hat{y})\frac{Y - \overline{y}}{2Y} = (v + \hat{y} - B)\frac{\underline{y} + Y}{2Y} \tag{2}$$
$$(v + \overline{y} - B)\frac{\underline{y} + Y}{2Y} = v + \overline{y} - A \tag{3}$$

Equilibrium (2)

- Solving (1)-(3) we obtain the thresholds.
  If we set $v = 0$, $Y = 2$, $A = 1$ and $B = -1$ they become

$$\underline{y} = \frac{1}{2}(3 - \sqrt{33}) \simeq 1.4$$
$$\hat{y} = 0$$
$$\overline{y} = \frac{1}{2}(\sqrt{33} - 3) \simeq -1.4$$

- Thus, in equilibrium the probability of a market buy/sell order next period is $\frac{2-1.4}{2\cdot 2} = \frac{-1.4+2}{2\cdot 2} = 0.15$
- Given a large spread ($A - B = 2$), limit order traders are willing to accept a low execution probability (15%) in order to obtain better price

Dynamic analysis: Conclusion

- Trade occurs when seller and buyer meet and agree on terms
  - Exchange brings sellers and buyers together
  - Impatient traders search more actively (use market orders)
  - Patient traders are more passive, offer liquidity (use limit orders)
- Ordinary traders can use limit orders:
  1. to compete with market makers (and profit from providing liquidity as a self-contained activity)
  2. or to reduce their trading costs by accepting some non-execution risk. E.g., **?**: avg trading cost of a large institutional trader is $\sim$ 0.015% (a couple of orders of magnitude smaller than quoted spreads), arguably due to using LOs to minimize price impact.
- Limit order book is dynamic: influence the choice of order
- In reality, many features to think about when placing orders

Dynamics with adverse selection

- Our analysis above focuses on non-execution risk, ignoring adverse selection
  - which can arise due to privately informed investors picking off the limit orders
  - or even public news arriving over time – then all future traders are more informed!
  - see ch. 6.4.3 & 6.6 of the textbook or the extra slides
- There is not too much interaction between the two, but there's an interesting dynamics of spread w.r.t. volatility of μ_t ($= v_t$ in the book)
- Read this at home.

Public information: issues and solutions

- **Monitoring costs**: if limit traders do not monitor the news constantly, their orders will get picked off. Monitoring reduces adverse selection, but is costly.
- **Pegged limit orders**: tied to another security or another market, automatically repriced when that price changes. Another insurance device, available on selected exchanges.
- **Algorithmic trading**: allows to automatically monitor relevant variables and reprice limit orders as needed. But at the same time increases the speed at which orders can be picked off, so cuts both ways.

Informed limit trading?

- But of course, the best way to avoid being picked off is to be already informed.
- Even the full version of Foucault model only deals with public info – there are no privately informed traders.
- But data shows that LOs are informative (**???**)
- Even more: informed traders might be using more LOs than MOs (**?**)
- **?** have a LOB model in which informed traders use both LOs and MOs, depending on how full the book is.

LOB: conclusion

- A lot of stuff going on in LOBs, difficult to analyze, but we tried.
- See **?** for a slightly more detailed review of literature on LOBs.

Exercise for next week

- Solve exercise 5 after chapter 6 (page 235) in the textbook on the effect of fees charged for limit orders and market orders in Foucault model

Dynamic model with adverse selection

- Consider now the Foucault model with public news: the market valuation evolves as $\mu_t = \mu_{t-1} + \epsilon_t$ where $\epsilon_t \in \{-\sigma, \sigma\}$ with equal probabilities.
- μ_t is observed in period t (but LOs from $t-1$ cannot be cancelled or repriced). The period-t trader has valuation $\mu_t + y_t$
- Again, we look for a stationary equilibrium. In particular:
  - Suppose that $A_t - \mu_{t-1} = \mu_{t-1} - B_t = S$ is constant.
  - Suppose there exist $\underline{y}(\epsilon_t), \overline{y}(\epsilon_t), \hat{y}(\epsilon_t)$ that describe the trader's strategy in *each* period as a function of the value innovation ϵ_t
  - Thus there exist time-invariant P_M^S and P_M^B that characterize the probability of a next-period market sell/buy order

Dynamic model with adverse selection (2)

- Intuitively, $\underline{y}(\epsilon_t), \overline{y}(\epsilon_t), \hat{y}(\epsilon_t)$ are decreasing in ϵ_t:
  - higher ϵ_t means more favorable quotes for MB (A_{t-1} is relatively low),
  - less favorable for MS (B_{t-1} is relatively high)
- The interesting trade-off here is generated by σ: when σ increases,
  - the worse is the risk of being picked-off (so limit orders are less appealing),
  - but the lower is the execution risk (so limit orders are more appealing)

Dynamic model with adverse selection (3)

- To solve the model: the indifference conditions at $\underline{y}, \hat{y}, \bar{y}$ are

$$
\begin{aligned}
B_t - (\mu_t + \underline{y}(\epsilon_t)) &= (A_{t+1} - \mathbb{E}_t[\mu_{t+1}|MB] - \underline{y}(\epsilon_t))P_M^B \\
(A_{t+1} - \mathbb{E}_t[\mu_{t+1}|MB] - \hat{y}(\epsilon_t))P_M^B &= (\mathbb{E}_t[\mu_{t+1}|MS] + \hat{y}(\epsilon_t) - B_{t+1})P_M^S \\
(\mathbb{E}_t[\mu_{t+1}|MS] + \overline{y}(\epsilon_t) - B_{t+1})P_M^S &= \mu_t + \overline{y}(\epsilon_t) - A_t
\end{aligned}
$$

- Focus on the first and last equation. Use the definition of S:

$$
\begin{aligned}
-S - \epsilon_t - \underline{y}(\epsilon_t) &= (S - \mathbb{E}_t[\epsilon_{t+1}|MB] - \underline{y}(\epsilon_t))P_M^B \\
(S + \mathbb{E}[\epsilon_{t+1}|MS] + \overline{y}(\epsilon_t))P_M^S &= -S + \epsilon_t + \overline{y}(\epsilon_t)
\end{aligned}
$$

- Solve for $\underline{y}(\epsilon_t)$ and $\overline{y}(\epsilon_t)$. Determine S from $A_t = \mathbb{E}_t[\mu_t|MB]$ and $B_t = \mathbb{E}_t[\mu_t|MS]$. Notice adv. sel. only affects limit orders. Back