

Financial Markets Microstructure

Lecture 14

Value of Liquidity
Chapter 9 of FPR

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Previously on FMM

- **Transparency** mostly reallocates welfare across market participants
 - Uninformed traders benefit from T, so T helps liquidity
 - Insiders may lose, so T worsens price discovery
 - Dealers/exchanges may win or lose
- But transparency may also impede risk sharing, foster collusion, and have adverse effects when it is asymmetrically distributed

Today: value of liquidity

- So far we looked at how illiquidity makes asset's trade price deviate from its fundamental value
- But illiquidity may itself affect the asset *value*
- Case study – [U.S. Treasury Notes and Bills](#): (Amihud and Mendelson [1991])
 - notes are long-term (2-10y), bills are short-term ($< 12\text{m}$) US govt loans
 - differ only in terms – so soon-to-mature notes are equivalent to bills
 - but notes trade at a discount relative to bills (i.e., offer higher returns) (as of 1991)
 - why? Notes are less liquid (larger spread and brokerage fees). Why less liquid though?

Value of liquidity

- Why does liquidity affect asset value?
- Intuitively, an illiquid asset is costlier to transact
 - Traders take into account transactions costs
 - Require a return that compensates for the cost
 - Liquidity premium: less liquid assets trade at lower prices
- Liquidity need not be constant over time
 - If illiquidity rises, asset price falls
 - If future liquidity is random, this is a risk factor
 - Liquidity risk may be priced

This lecture:

- 1 Toy model
- 2 Clientelle effects
- 3 Liquidity risk
- 4 Arbitrage

Liquidity premium (Amihud and Mendelson [1986])

- Before, traders cared only about fundamental value. In this model they care about resale value.
- Consider an asset with *constant* relative spread, $s = (a_t - b_t)/m_t$, but fluctuating midprice m_t

■ Note that

$$a_t = m_t \left(1 + \frac{s}{2}\right) \text{ and } b_t = m_t \left(1 - \frac{s}{2}\right)$$

- Consider a trader who plans to:

- 1 buy at t , at the respective ask price a_t ,
- 2 hold the asset for h periods, and sell at b_{t+h} .

To simplify, suppose the asset pays no dividends.

Deriving the premium: risk-adjusted return

- Let r denote the risk-adjusted **real return** per period **required by the market**. What? Then

$$a_t = \frac{\mathbb{E}(b_{t+h})}{(1+r)^h}, \quad \Rightarrow m_t = \frac{\mathbb{E}(m_{t+h})}{(1+r)^h} \times \frac{1 - \frac{s}{2}}{1 + \frac{s}{2}}$$

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- If we estimate the required return r using mid-quotes, there is a bias due to illiquidity.
- Let R be the **nominal return** rate, estimated from the midquotes:

$$m_t = \frac{\mathbb{E}(m_{t+h})}{(1+R)^h}$$

The observed R is different from r !

Deriving the premium: approximation

- Thus, we have

$$(1 + R)^h = (1 + r)^h \times \frac{1 + \frac{s}{2}}{1 - \frac{s}{2}} \quad (9.5)$$

Thus: (see next slide for derivation)

$$R \simeq r + \frac{s}{h}$$

- Essentially, the asset's return needs to be higher by s/h in order to compensate for the liquidity cost
 - The difference $R - r$ is a **liquidity premium**
 - Take h as representative trader's holding period for asset

Appendix on the approximation

- To get the approximation of the previous slide, we must use the approximation $\ln(1+x) \simeq x$ for small x
- Recall further that $\ln x^h = h \ln x$
- So taking logs of both sides of $(1+R)^h = (1+r)^h \times \frac{1+\frac{s}{2}}{1-\frac{s}{2}}$, we get

$$h \ln(1+R) = h \ln(1+r) + \ln\left(1+\frac{s}{2}\right) - \ln\left(1-\frac{s}{2}\right)$$

and assuming r , R , and s are small we apply the approximation

$$hR \simeq hr + \frac{s}{2} - \left(-\frac{s}{2}\right)$$

- Rearranging the terms, we get $R \simeq r + \frac{s}{h}$.

- Empirical evidence confirms *positive* liquidity premium for stocks, bonds
- For a more general model and empirics, see Bongaerts, De Jong, and Driessen [2011]

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Clientelle effects

- We obtained $R = r + s/h$ in our toy model
- In reality investors differ in h , expected holding period
- Consider a toy extension of our toy model, with:
 - Two types of investors with $h_1 < h_2$
 - Two assets with $s_1 < s_2$



Clientelle effects

- Suppose in eqm h_1 -investors trade in s_1 -asset and h_2 -investors trade in s_2 -asset
- For this to be an eqm, need $R_1 - s_1/h_1 \geq R_2 - s_2/h_1$ and $R_2 - s_2/h_2 \geq R_1 - s_1/h_2$
- The two conditions are equivalent to

$$\frac{1}{h_2} \leq \frac{R_2 - R_1}{s_2 - s_1} \leq \frac{1}{h_1} \quad (9.10)$$

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- There exist R_1, R_2 (and r) which solve this so all ok
 - There would not be a solution if we assumed the opposite kind of separation
 - We also cannot have both groups indifferent between both assets (would need two equalities in 9.10)

Clientelle effects: discussion

- Some investors specialize in illiquid assets / hope to earn the liquidity premium
 - Should in equilibrium be those who trade less frequently
- (would this explain the case of Treasury Bills vs Bonds?)
- Note: more adverse selection implies larger spread, hence attracts traders with large h

Clientelle effects: discussion

- Some investors specialize in illiquid assets / hope to earn the liquidity premium
 - Should in equilibrium be those who trade less frequently
- (would this explain the case of Treasury Bills vs Bonds?)
- Note: more adverse selection implies larger spread, hence attracts traders with large h
- We assumed that h are fixed, but all the same logic applies if h is random (e.g., traders randomly get liquidity shocks).
- Clientelle effects would then apply whenever different groups of traders have different *distributions* of h .

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Liquidity risk

- IRL, spread s randomly fluctuates over time
- Further, liquidity of any given asset may be arbitrarily correlated with that of other assets or the whole market
- These are risk factors which can also be priced
- Use the Liquidity CAPM model of Acharya and Pedersen [2005]

reminder: regular CAPM

- The standard CAPM postulates that return r_j on asset j is governed by the risk-free rate r_f and a risk premium, which depends on the correlation of r_j with the market return r_M :

$$\mathbb{E}[r_j] = r_f + \beta_j [\mathbb{E}[r_M] - r_f]$$

$$\text{with } \beta_j = \frac{\mathbb{C}(r_j, r_M)}{\mathbb{V}(r_M)}$$

- In particular, only *systematic* risk enters asset price
- *Idiosyncratic* risk of the asset can be diversified away

Liquidity CAPM

- Investors care for net return $r = R - s$ where s now denotes the liquidity premium
 - Let f denote risk-free, M the market
- Plugging these into the CAPM equation, we get

$$\mathbb{E}[R_j - s_j] = r_f + \lambda_M \beta_j$$

where $\lambda_M = \mathbb{E}[R_M - s_M] - r_f$ is the market risk premium and

$$\beta_j = \frac{\mathbb{C}(R_j - s_j, R_M - s_M)}{\mathbb{V}(R_M - s_M)}$$

■ Expand $\mathbb{C}(R_j - s_j, R_M - s_M)$:

$$= \mathbb{C}(R_j, R_M) + \mathbb{C}(s_j, s_M) - \mathbb{C}(R_j, s_M) - \mathbb{C}(s_j, R_M)$$

to get $\beta_j = \beta_{1j} + \beta_j^{liq} = \beta_{1j} + \beta_{2j} - \beta_{3j} - \beta_{4j}$ with

$$\beta_{1j} = \frac{\mathbb{C}(R_j, R_M)}{\mathbb{V}(R_M - s_M)} \quad : \text{ordinary } \beta$$

$$\beta_{2j} = \frac{\mathbb{C}(s_j, s_M)}{\mathbb{V}(R_M - s_M)} \quad : \text{hedge liquidity with liquidity}$$

$$\beta_{3j} = \frac{\mathbb{C}(R_j, s_M)}{\mathbb{V}(R_M - s_M)} \quad : \text{hedge liquidity with returns}$$

$$\beta_{4j} = \frac{\mathbb{C}(s_j, R_M)}{\mathbb{V}(R_M - s_M)} \quad : \text{hedge returns with liquidity}$$

Liquidity risk: alternative model

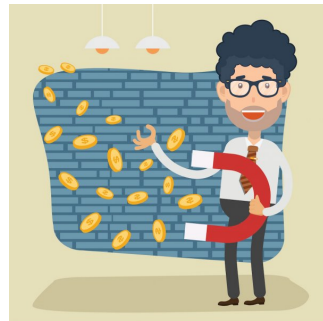
- Earlier in the course we derived the spread given the asset return/value (Glosten-Milgrom)
- Earlier today we did the opposite: derived the required return keeping the spread fixed (Amihud-Mendelson)
- Duffie, Gârleanu, and Pedersen [2005, 2007] do both simultaneously! In their model:
 - Traders are randomly hit with liquidity shocks, so want to buy/sell asset over time depending on their current situation
 - But may not always find a trade – can get stuck with asset when shocked/without asset when not shocked
 - This liquidity risk enters the equilibrium asset price
 - (And the spread is driven by dealers' market power)
- Click [here](#) if you want to see the model (you do not need to know it for the exam)

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Arbitrage

- Main tenet of economics and finance: **no arbitrage**
 - Assets that generate same cash flows must cost the same
 - If arbitrage is possible, it is immediately exploited and then there is no more arbitrage.



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- **Bliz quiz:** how does this relate to our situation? (think bills-vs-bonds)
 - 1 Arbitrage opportunities exist because arbitrage is costly to exercise
 - 2 Arbitrage opportunities exist because all of economics (except our course) is wrong
 - 3 No arbitrage opportunities exist



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 - 1 Arbitrage opportunities exist because arbitrage is costly to exercise
↑ **textbook answer**
 - 2 Arbitrage opportunities exist because all of economics (except our course) is wrong
 - 3 No arbitrage opportunities exist ← **my answer**



Arbitrage

- Main tenet of economics and finance: **no arbitrage**
 - Assets that generate same cash flows must cost the same
 - If arbitrage is possible, it is immediately exploited and then there is no more arbitrage.
- Depending on semantics:
 - **arbitrage in nominal prices** cannot be realized due to limited liquidity...
 - (also, arbitrage itself is costly, due to leverage and short-selling constraints)
 - ...so there is **no real arbitrage**



Conclusion

- Empirical research finds evidence on both a **liquidity premium** and a **liquidity risk premium** on stocks
- Further, overall market liquidity may vanish at crisis times when asset prices drop rapidly
 - Important risk for investors, especially speculators
 - Financial institutions are required to hold robustly liquid assets
 - In general, risky positions require costly collateral, margins

Exercise

Ex.1 from ch.9 (p.347)

References I

- V. V. Acharya and L. H. Pedersen. Asset pricing with liquidity risk. *Journal of Financial Economics*, 77(2):375–410, Aug. 2005. ISSN 0304-405X. URL <https://doi.org/10.1016/j.jfineco.2004.06.007>.
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References II

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Required return

- We have been living in a world with only one asset. In reality, assets “compete” for investors’ attention.
- In market equilibrium, risk-adjusted returns are equalized across assets.
- The resulting “market” return r is what investors can get by investing in any asset, and any new asset must generate return (at least) r to attract funds.

[Back](#)



Duffie, Gârleanu, and Pedersen [2005, 2007] Model

- One asset:
 - Pays dividends to its holders each period
 - Traders can hold **one** or **zero units** of the asset
 - The alternative is a bank which pays interest r
 - Assume the asset is supplied to fraction $q < 1/2$ of the population
- Unit mass (continuum) of traders, each has either high or low value for the dividend:
 $j = h, l$
 - If $j = h$, value today's dividend at 1 (high value traders); if $j = l$ then value dividend at $1 - c$ (low value traders), where $c \in (0, 1)$
 - Every period, fraction ψ of traders switch (from h to l or vice versa)
 - (In the long run, half the traders have high value)
 - Switchers would like to trade the asset (h to $l \rightarrow$ want to sell; l to $h \rightarrow$ want to buy)
 - Those willing to trade search for a dealer, find one with probability $\phi < 1$

DGP Model (2)

- Exact timing within a period:
 - 1 investor receives dividend payoff (\$1 or $\$(1 - c)$)
 - 2 valuation changes w.p. ψ
 - 3 if needed, trader looks for a dealer; a match happens w.p. ϕ
- Dealers have some bargaining power because they are hard to find
 - Rejecting a dealer's quote means the trader has to wait one period for a chance to meet another dealer – costly (and risky) for a trader
 - Use standard notation: dealers quote ask a and bid b ; spread is $S = a - b$; midprice is $\mu = (a + b)/2$
- Look for a stationary equilibrium with a and b constant over time, and all traders with trading needs (“ h without an asset” and “ l with an asset”) looking for a dealer, and agreeing to trade at a and b respectively iff it is optimal for them (may mix).

Solving the DGP model

- Since $q < 1/2$, not all buyers get to buy (buyers must be indifferent \iff dealers have all market power)
- Denote \bar{a} = max price that a buyer will pay, and \bar{b} = min price that a seller will accept
- **Buy side:** limited supply \rightarrow mixed equilibrium. Set $a = \bar{a}$ and set buy probability conditional on finding dealer (can choose this since buyers are indifferent) to

$$p^B = \phi \cdot \frac{\pi_S}{\pi_B},$$

π_S : fraction willing to sell; π_B : fraction willing to buy

- **Sell side:** no restraint, sell with prob. 1, bargain with the dealer over surplus:

$$b = z\bar{b} + (1 - z)\mu.$$

Solving the DGP model (2)

- Method: identify \bar{a} and \bar{b} , and then solve for a and b
- Denote the present discounted cash-flow value (as of beginning of the period) of an asset owner with private valuation j by V_j^o and that of a non-owner by V_j^{no}
- Since non-owner buys if $V_h^o - a \geq V_h^{no}$ and owner sells if $V_l^{no} + b \geq V_l^o$, we get

$$\bar{a} = V_h^o - V_h^{no}, \quad (1)$$

$$\bar{b} = V_l^o - V_l^{no} \quad (2)$$

- So $a = V_h^o - V_h^{no}$ and $b = z(V_l^o - V_l^{no}) + (1 - z)\mu$.
- Finally, we must calculate the value functions

Solving the DGP model (3)

- We will just look at two functions, the rest are similar

$$V_h^o = \frac{1}{1+r} + \frac{(1-\psi)V_h^o}{1+r} + \frac{\psi(1-\phi)V_l^o}{1+r} + \frac{\psi\phi(V_l^{no} + b)}{1+r}$$
$$V_h^{no} = \frac{\psi V_l^{no}}{1+r} + \frac{(1-\psi)(1-p^B)V_h^{no}}{1+r} + \frac{(1-\psi)p^B(V_h^o - \bar{a})}{1+r}$$

- Calculate V_l^o and V_l^{no} . Plug back into (1)-(2) to solve for \bar{a} and \bar{b}

DGP: Results (1)

- Ask price is then

$$a = \frac{1}{r} - \frac{2\psi}{r(1+z)} \left(1 - \phi \frac{1-z}{2} \right) S,$$

where S is the spread

$$S = a - b = \frac{(1+z)c}{2(r+2\psi) + (1-2\psi)\phi(1-z)}$$

- If $\psi > 0$ the ask price is less than the $1/r$ which would arise if it were always (efficiently) held by high-value traders
- This is due to liquidity costs: buyers anticipate that they will sell in the future and incur the spread cost

DGP: Results (2)

- The midquote is

$$\mu = \frac{1}{r} \left(1 - \frac{c}{2} \right) - \frac{c}{2r} \cdot \frac{\phi(1-z)}{2r + 4\psi - (2\psi - 1)\phi(1-z)}$$

- Increasing in ϕ : larger ϕ = smaller prob of not finding a trade ϕ = smaller illiquidity risk
= more valuable asset