

Financial Markets Microstructure

Lecture 5

Glosten-Milgrom Model

Chapter 3.3 of FPR

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What did we do last time?

- 1 Argued that market thinness is not the only source of illiquidity
- 2 Poked holes in the Efficient Market Hypothesis
- 3 Defined price efficiency in many ways
- 4 Began talking about the GM model

Today

1 more Glosten-Milgrom!

This lecture:

1 Glosten-Milgrom (reviewed and continued)

2 GM example

3 GM: conclusions

4 extras

GM85: Overview

- Dynamic model, periods $t = 1, 2, \dots$;
(though we will be analyzing the stage game for a given period – essentially static)
- Two players in every period:
 - trader and dealer
 - **dealer** long-lived; trader new every period
 - **trader** can be informed or not
- One asset with fundamental value v (unknown), common belief $v \sim F(v)$

GM85: Model (1)

Trader: is either a speculator or a noise trader, can submit a market order $d_t \in \{1, -1\}$ to buy or sell one unit of the asset with fundamental value v (or do nothing, $d_t = 0$)

- **Speculator** (probability π): has private information about v .
 - We will usually assume speculator simply knows v (not much changes if he only has a noisy private signal about it).
 - Risk neutral, chooses his market order d_t to maximize expected profit $d_t \cdot (v - p_t)$:
- **Noise trader** (probability $1 - \pi$): no pvt info about v ; trades for other reasons (hedging, liquidity).
 - We assume he follows some fixed strategy: buys with probability β_B ; sells w.p. β_S ; abstains w.p. $1 - \beta_B - \beta_S$
 - **Important:** this assumption is for simplicity only; this strategy can be perfectly rational! We just don't model what generates it.
E.g., could say noise traders choose d_t to maximize profit $\mathbb{E}[d_t(v + y_t - p_t)]$, where y is t -trader's idiosyncratic valuation (due to risk, time, liquidity preferences...)

GM85: Model (2)

Dealer (market maker)

- Risk neutral
- Willing to trade **exactly one unit** (buy/sell/no trade) each period
- Sets **bid and ask prices** (for a single unit)
- Quote price before seeing trade (limit order)
- Does not know whether trader is speculator or noise trader (but knows π)
- Expected profit from trade is $\mathbb{E}[-d_t(v - p_t)]$
- **Competitive**: prices=expected asset value conditional on information
- Trading is sequential: market orders served one by one

■ Equilibrium:

- An equilibrium consists of **bid and ask prices** and **speculator's strategy**
- They must be such that: (i) prices are competitive (zero profit for MM), (ii) speculator best-responds to prices (maximizes expected gain).

Analysis. A: Market making

- Dealer quotes bid and ask prices on *one unit*
 - Can revise prices between each incoming trade
- Quoted ask price a_t only relevant if next incoming trader decides to buy
 - Dealer's payoff in this case is given by $\mathbb{E}[a_t - v | \Omega_{t-1}, Buy] = a_t - \mathbb{E}[v | \Omega_{t-1}, Buy]$
 - Same for bid b_t ; payoff: $\mathbb{E}[v | \Omega_{t-1}, Sell] - b_t$
 - (Note payoffs above rely on risk-neutrality)
- Perfect competition among dealers implies zero expected profit from either trade type \Rightarrow ask price and bid price are

$$a_t = \mathbb{E}[v | \Omega_{t-1}, Buy];$$

$$b_t = \mathbb{E}[v | \Omega_{t-1}, Sell].$$

- Notice that both sides of the equality depend on prices

Analysis. B: Informed trading

- Speculator knows v . Given prices a_t and b_t , the expected profits Π are:

$$\Pi(v, a_t, b_t, d_t) = \begin{cases} v - a_t & \text{if } d_t = 1; & (\text{Buy}) \\ 0 & \text{if } d_t = 0; & (\text{Abstain}) \\ b_t - v & \text{if } d_t = -1. & (\text{Sell}) \end{cases}$$

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- Speculator's best response to (a_t, b_t) is: (assume $a_t \geq b_t$)

- Buy when $v > a_t$, i.e. when v is large enough
- Sell when $v < b_t$, i.e. when v is small enough
- Abstain if $a_t > v > b_t$

Analysis. C: Equilibrium definition

Dealer must make zero profit (competition), traders must trade optimally. This gives us two **equilibrium conditions**.

- Let σ_t denote the speculator's strategy, where $\sigma_t(d_t|v)$ is the probability that the speculator places order d_t if value is v
- An **equilibrium** consists of **prices** (a_t, b_t) and **strategy** σ_t such that:

1 the ask and bid prices solve

$$a_t = \mathbb{E}[v|\Omega_{t-1}, Buy];$$

$$b_t = \mathbb{E}[v|\Omega_{t-1}, Sell],$$

given σ_t

2 for each v , σ_t solves

$$\max_{\sigma_t} \{ \sigma_t(1|v)[v - a_t] + \sigma_t(-1|v)[b_t - v] \},$$

given (a_t, b_t) .

GM Example

- **Single period:** Suppose only one period (drop t subscript, drop Ω)
- **Binary outcome:** $v \in \{0, 1\}$, equally likely ex ante: $\mathbb{P}(v = 1) = 0.5$.
- Suppose $0 < b < a < 1$ and noise trader's order obeys $\beta_B = \beta_S = 0.5$.

Questions:

- 1 What is the speculator's trading strategy?
- 2 Can you derive dealer's **prices a and b** , as a function of π ?
 - If not, refresh your knowledge of conditional expectations and try again.
 - If you already read the solution in the book, try to replicate it without looking back at the book.
- 3 Are the resulting prices efficient? (Check all three forms)

Analysis. D: Solving for equilibrium (1)

- The orders reveal information about v . E.g., a **buy order** is submitted:
 - either by a noise trader (probability $(1 - \pi)\beta_B$) – no new information, $\mu_t = \mu_{t-1} = \mathbb{E}[v|\Omega_{t-1}]$;
 - or by a speculator (probability $\pi\mathbb{P}(v \geq a_t|\Omega_{t-1})$) – then learn that $v \geq a_t$, so $\mu_t = \mathbb{E}[v|\Omega_{t-1}, v \geq a_t]$.
- Then a_t is given by (using Bayes' rule and law of total probability; N=Noise, I=Informed):

meaning that in the end, $a_t \geq \mu_{t-1}$.

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- Then a_t is given by (using Bayes' rule and law of total probability; N=Noise, I=Informed):

$$\begin{aligned} a_t &= \mathbb{E}[v|\Omega_{t-1}, Buy] \\ &= \mathbb{P}(N|\Omega_{t-1}, Buy) \cdot \mathbb{E}[v|\Omega_{t-1}, Buy, N] + \mathbb{P}(I|\Omega_{t-1}, Buy) \cdot \mathbb{E}[v|\Omega_{t-1}, Buy, I] \\ &= \frac{\mathbb{P}(Buy, N|\Omega_{t-1})}{\mathbb{P}(Buy|\Omega_{t-1})} \cdot \mathbb{E}[v|\Omega_{t-1}] + \frac{\mathbb{P}(Buy, I|\Omega_{t-1})}{\mathbb{P}(Buy|\Omega_{t-1})} \cdot \mathbb{E}[v|\Omega_{t-1}, v \geq a_t] \\ &= \frac{(1 - \pi)\beta_B}{(1 - \pi)\beta_B + \pi\mathbb{P}(v \geq a_t)} \cdot \mu_{t-1} + \frac{\pi\mathbb{P}(v \geq a_t|\Omega_{t-1})}{(1 - \pi)\beta_B + \pi\mathbb{P}(v \geq a_t)} \cdot \mathbb{E}[v|\Omega_{t-1}, v \geq a_t], \end{aligned}$$

meaning that in the end, $a_t \geq \mu_{t-1}$.

Analysis. D: Solving for equilibrium (2)

- Similarly for **sell orders**:
 - sell order from a noise trader arrives with [unconditional] probability $(1 - \pi)\beta_S$ – no new information, $\mu_t = \mu_{t-1}$;
 - sell order from a speculator arrives with probability $\pi\mathbb{P}(v \leq b_t | \Omega_{t-1})$ – then learn that $v \leq b_t$, so $\mu_t = \mathbb{E}[v | \Omega_{t-1}, v \leq b_t]$.
- Then b_t is given by:
 - so $b_t \leq \mu_{t-1}$, and we have confirmed that indeed $a_t \geq b_t$.

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- Then b_t is given by:

$$\begin{aligned} b_t &= \mathbb{E}[v | \Omega_{t-1}, \text{Sell}] \\ &= \frac{(1 - \pi)\beta_S}{\mathbb{P}(\text{Sell} | \Omega_{t-1}, v)} \cdot \mathbb{E}[v | \Omega_{t-1}] + \frac{\pi\mathbb{P}(v \leq b_t | \Omega_{t-1})}{\mathbb{P}(\text{Sell} | \Omega_{t-1}, v)} \cdot \mathbb{E}[v | \Omega_{t-1}, v \leq b_t] \end{aligned}$$

- where $\mathbb{P}(\text{Sell} | \Omega_{t-1}, v) = (1 - \pi)\beta_S + \pi\mathbb{P}(v \leq b_t)$,
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- so $b_t \leq \mu_{t-1}$, and we have confirmed that indeed $a_t \geq b_t$.
- These prices a_t, b_t together with the speculator's trading strategy constitute the equilibrium.

Analysis. E: Profits

- **Informed traders** earn positive profit (since know v and have an option of doing nothing)
- Dealers assumed competitive, hence zero profit
- **Uninformed traders** incur a loss
 - Although this is because we modeled this as a zero-sum game and explicitly ignored the uninformed traders' potential trading motives.

Analysis. F: Price efficiency

- $\mu_t \equiv \mathbb{E}[v|\Omega_t]$ is the expectation of v *after* the time- t trade order is observed (by the dealer and outside observers). Note that in our model:

$$\mu_t = \begin{cases} a_t > \mu_{t-1} & \text{if buy order at } t; \\ b_t < \mu_{t-1} & \text{if sell order at } t. \end{cases}$$

- Meaning market price is efficient: $p_t = \mu_t$
 - in **semi-strong** form (prices anticipate and incorporate all information conveyed through trades),
 - not in the **strong** form (which would be equivalent to $p_t = v$)

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- Meaning market price is efficient: $p_t = \mu_t$
 - in **semi-strong** form (prices anticipate and incorporate all information conveyed through trades),
 - not in the **strong** form (which would be equivalent to $p_t = v$)
- This is because dealers are **competitive** – dealers' market power would ruin efficiency

This lecture:

1 Glosten-Milgrom (reviewed and continued)

2 GM example

3 GM: conclusions

4 extras

Example (as in the book)

- Let's do an example similar to your homework; calculate the prices more explicitly.
- **Single period:** Suppose only one period (drop t subscript, drop Ω)
- **Binary outcome:** $v \in \{v^H, v^L\}$, with prior $\theta = \mathbb{P}(v^H)$
- **Prior value:** What is the prior value of the asset before trading?

$$\mu = \theta v^H + (1 - \theta) v^L.$$

Skip example

Example (2)

- How do we solve the model? Look for equilibrium with trade.
- Suppose $v^L < b < a < v^H$.
- Then speculator buys if $v = v^H$, sells if $v = v^L$.
 - That is, $\sigma(1|v^H) = 1$ and $\sigma(-1|v^L) = 1$
- The procedure is then the following
 - 1 Use the equilibrium conditions from before to calculate prices given the above speculator strategy
 - 2 Check that these prices satisfy $v^L < b < a < v^H$

Example (3)

- Let's solve for the ask price. First:

$$\mathbb{P}(Buy|v^H) = (1 - \pi)\beta_B + \pi$$

$$\mathbb{P}(Buy|v^L) = (1 - \pi)\beta_B$$

Then by Bayes' Rule

$$\begin{aligned}\mathbb{P}(v^H|Buy) &= \frac{\mathbb{P}(v^H)\mathbb{P}(Buy|v^H)}{\mathbb{P}(Buy)} \\ &= \frac{\theta[(1 - \pi)\beta_B + \pi]}{(1 - \pi)\beta_B + \pi\theta} \\ &= \theta + \frac{\theta(1 - \theta)\pi}{(1 - \pi)\beta_B + \pi\theta}\end{aligned}$$

Example (4)

- The ask price is the expected value, given a buy order:

$$\begin{aligned}a &= \mathbb{P}(v^H | Buy) v^H + [1 - \mathbb{P}(v^H | Buy)] v^L \\&= \left[\theta + \frac{\theta(1-\theta)\pi}{(1-\pi)\beta_B + \pi\theta} \right] v^H + \left[1 - \left(\theta + \frac{\theta(1-\theta)\pi}{(1-\pi)\beta_B + \pi\theta} \right) \right] v^L \\&= \mu + \frac{\theta(1-\theta)\pi}{(1-\pi)\beta_B + \pi\theta} (v^H - v^L).\end{aligned}$$

- Doing a similar exercise for b we find

$$b = \mu - \frac{\theta(1-\theta)\pi}{(1-\pi)\beta_S + \pi(1-\theta)} (v^H - v^L)$$

- Finally, we must check that our assumption holds: easy to check that $v^H > a > b > v^L$.
Hence, **this is an equilibrium**

Example: Lessons

$$a - \mu = \frac{\theta(1 - \theta)\pi}{(1 - \pi)\beta_B + \pi\theta}(v^H - v^L)$$
$$\mu - b = \frac{\theta(1 - \theta)\pi}{(1 - \pi)\beta_S + \pi(1 - \theta)}(v^H - v^L)$$

- Add the two expressions to get bid-ask spread $S = a - b$
 - S increases in π : more informed trading exacerbates adverse selection. Opposite for $(\beta_B + \beta_S)$.
 - If $\beta_B = \beta_S = 1/2$, S is increasing in $\theta(1 - \theta)$, i.e. spread higher when dealer faces greater initial uncertainty about v . Same for $(v^H - v^L)$.

Example: Price discovery

- Return to the multiperiod setting. One unit traded every period, v persistent.
- Trade flow is **informative** – trades have long-lasting effect on prices
- Each order conveys information, dealers learn, and

$$p_t \rightarrow v$$

Dynamics

prices **strong-form efficient** in the long run

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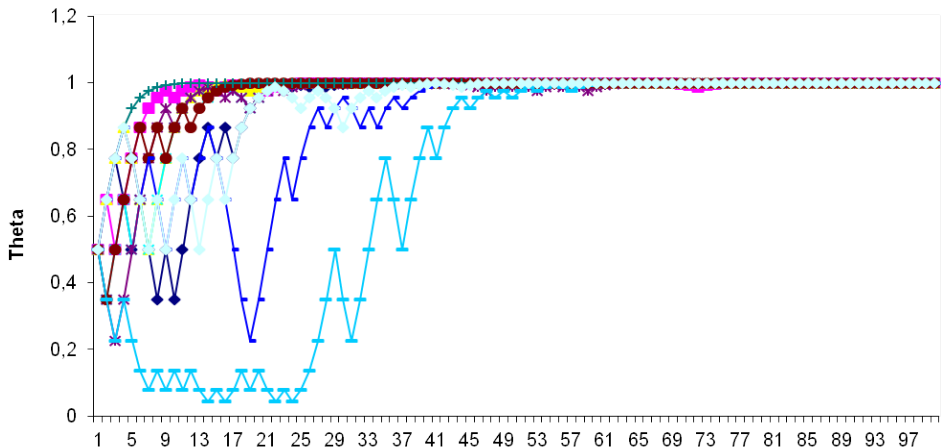
Dynamics

prices **strong-form efficient** in the long run

- Speed of price discovery increasing in π
 - Trade-off between **price discovery** and **liquidity**

Example: Simulation

Dealer beliefs: Each curve shows the evolution of dealer's beliefs in each run (10 runs of 100 orders)



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1 Glosten-Milgrom (reviewed and continued)

2 GM example

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GM: Summary of the findings

What did we learn from the Glosten and Milgrom model?

1 Information, prices and the spread

- Prices will reflect the information revealed by trades
- The spread is increasing in informational asymmetry (adverse selection) and in uncertainty about asset value

2 Informational efficiency

- Prices are always semi-strong efficient, in the long run also strong-form efficient

3 Noise trading

- Noise trading keeps the market liquid and improves spreads
- Informed speculation increases spreads, but improves price discovery - dilemma for regulators

(anti?)Features

- **Dealer model:** Prices are set each period, discriminative, normally competitive (zero profits)
- **Non-market clearing:** Only one unit traded - not market clearing (traders may wish to buy/sell more)
- Only **fundamental value** matters, no speculation/resale

Discussion

- **Insights:** Adverse selection as a driver of the spread
- **Shortcomings:** Trade fixed amount, trade once, no resale
- **Advantages:** (Relatively) simple analysis, flexible

Homework

■ Reading:

- Read two articles on absalon on how ESMA restricted trading and binary options and SEC restricted trading in certain stocks.
- What is the difference between the underlying assets in the two cases?
- Explain ESMA's decision using GM model.

■ Solving:

- FPR chapter 3, exercise 3 (GM model where speculators are not perfectly informed, but instead receive a signal about the value of the asset)
- GM example with $v \sim U[0, 1]$ (rest as in the problem assigned before today; goal: derive the equilibrium bid and ask prices)

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Suppose we are in the simple binary model with the following parameters

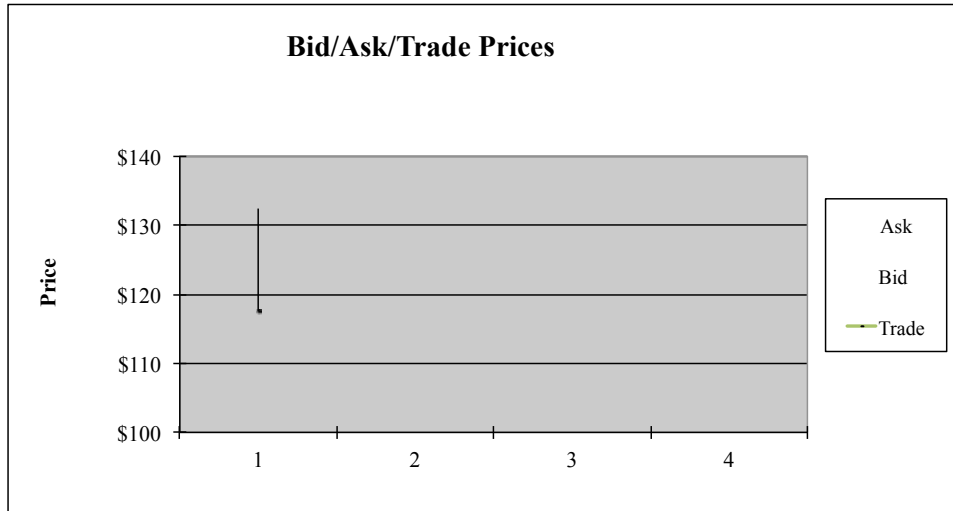
- Probability of informed speculators: $\pi = 0.3$
- Probability (ex ante) of high value: $\theta = 0.5$
- $v^H = 150$ and $v_L = 100$

Consider 12 periods, with the following sequence of buys (b) and sells (s)

ssbssssssssss

Dynamics

First period: sell



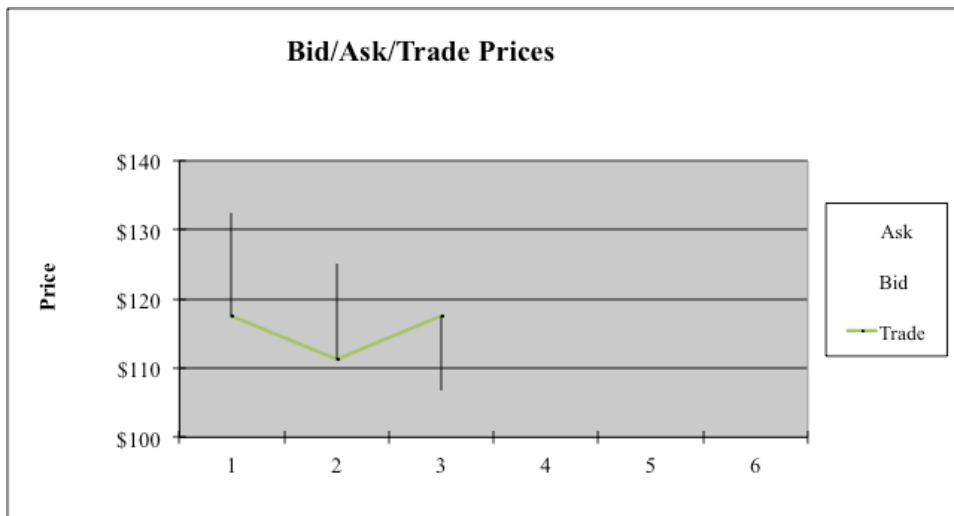
Dynamics

Second period: sell



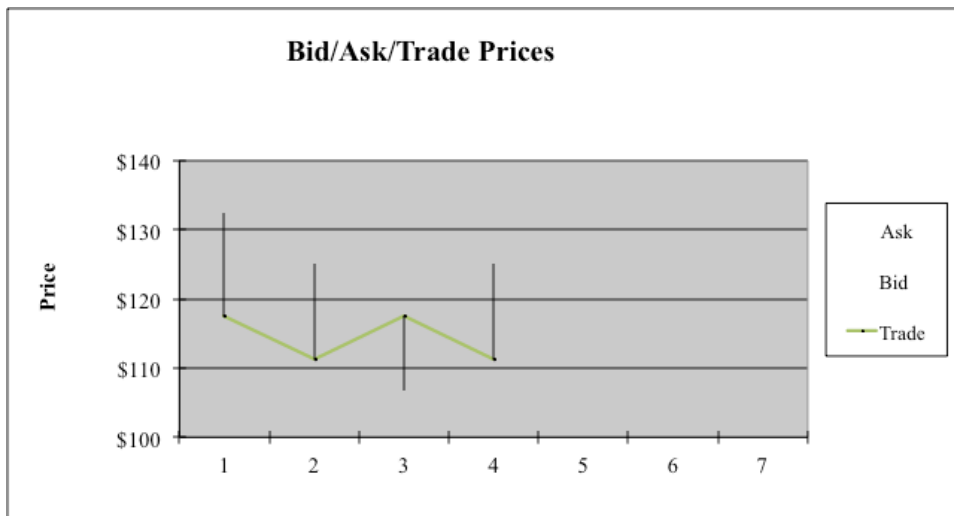
Dynamics

Third period: buy



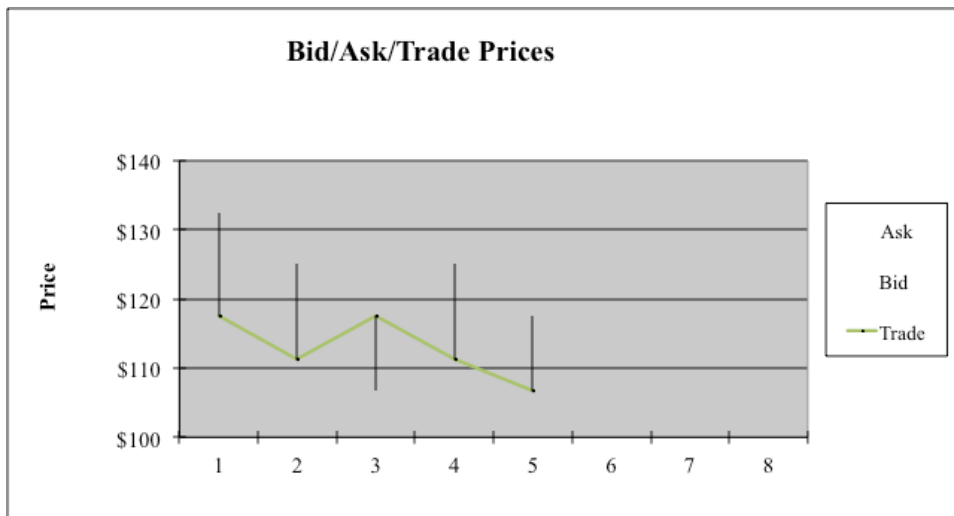
Dynamics

Fourth period: sell



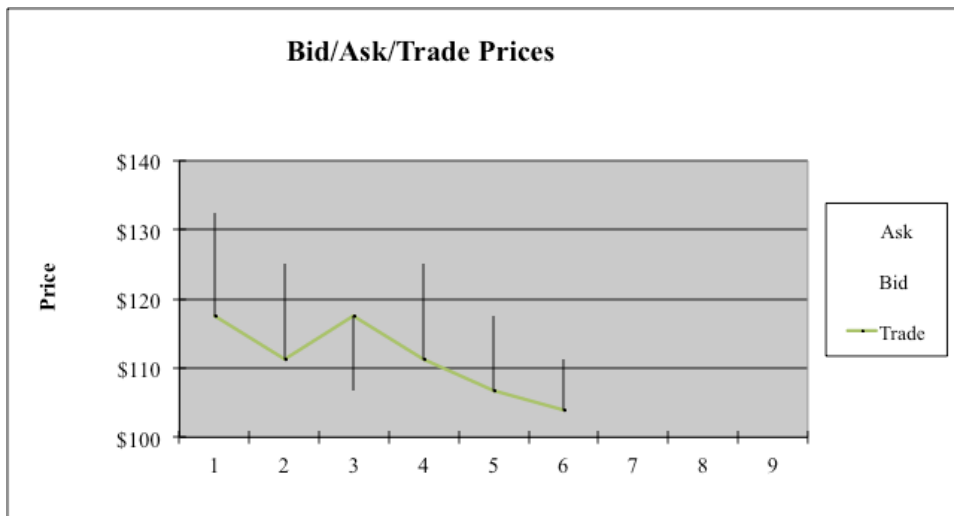
Dynamics

Fifth period: sell



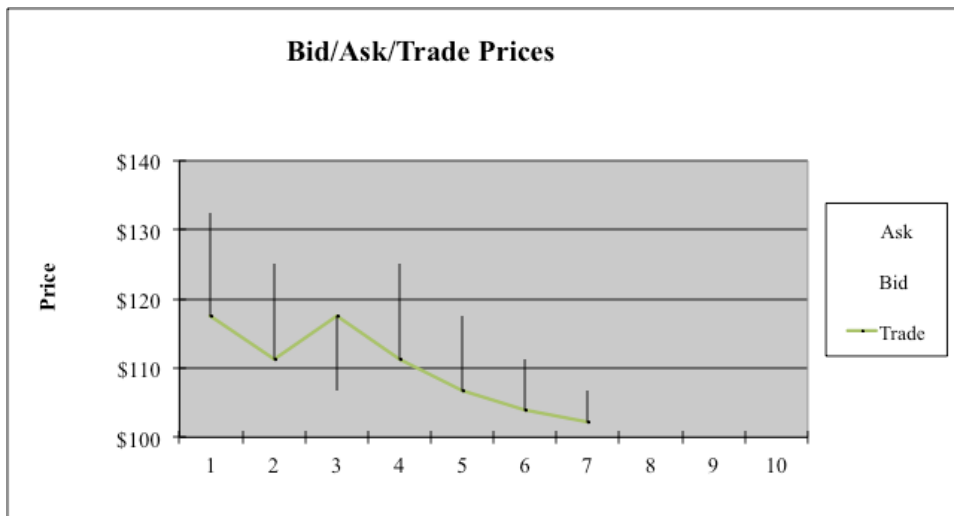
Dynamics

Sixth period: sell



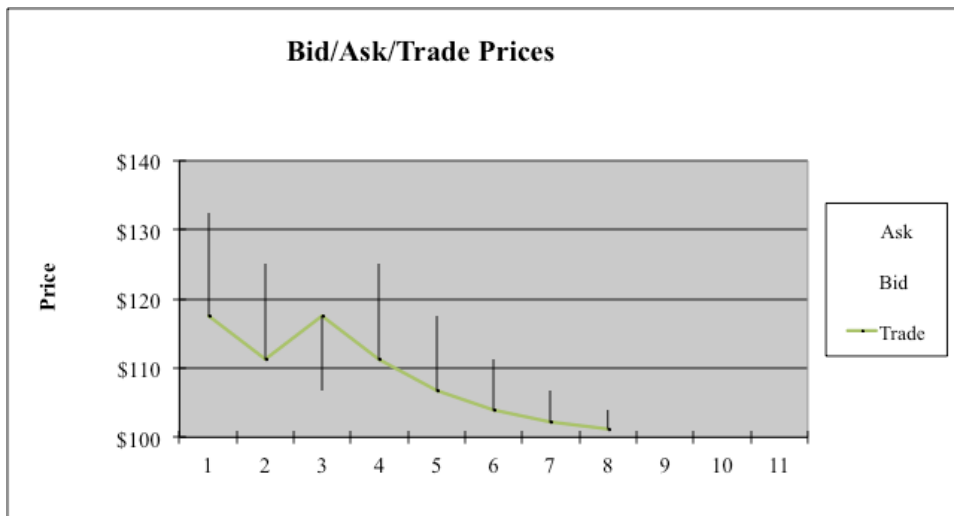
Dynamics

Seventh period: sell



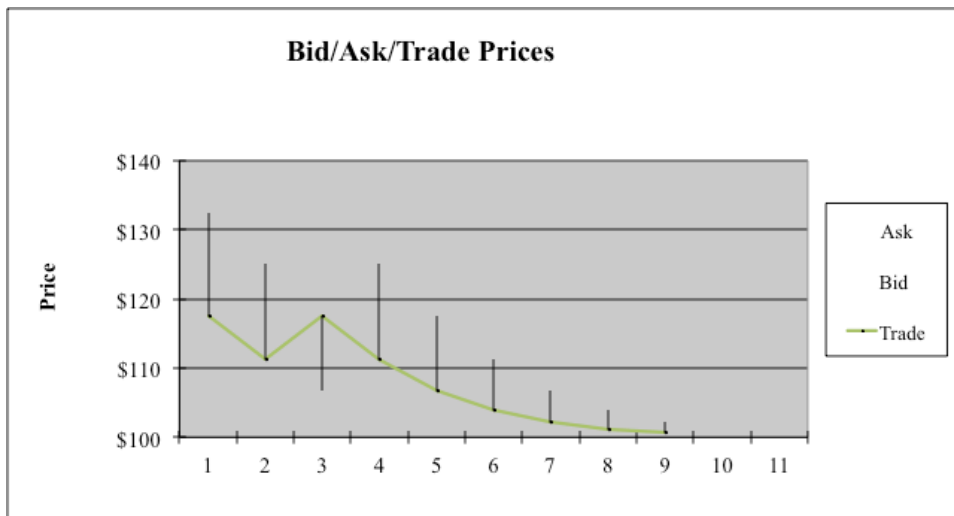
Dynamics

Eighth period: sell



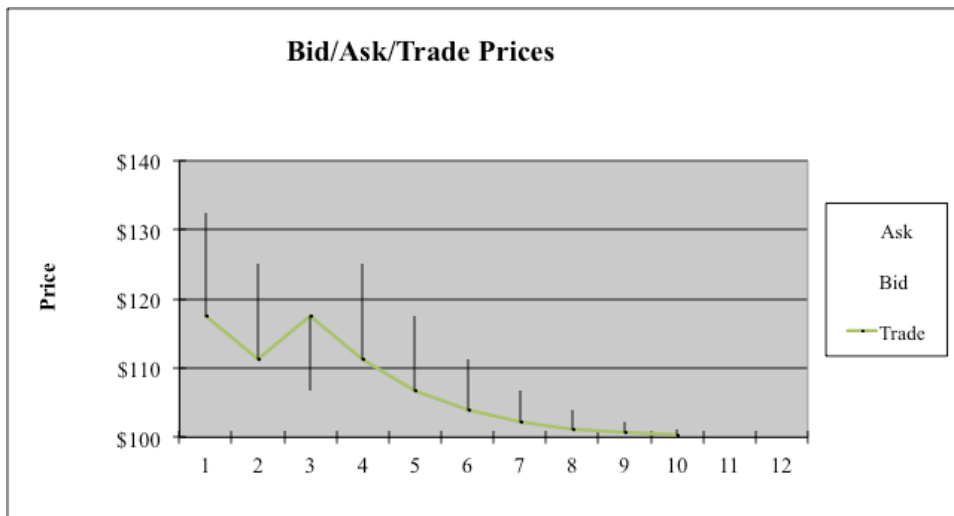
Dynamics

Ninth period: sell



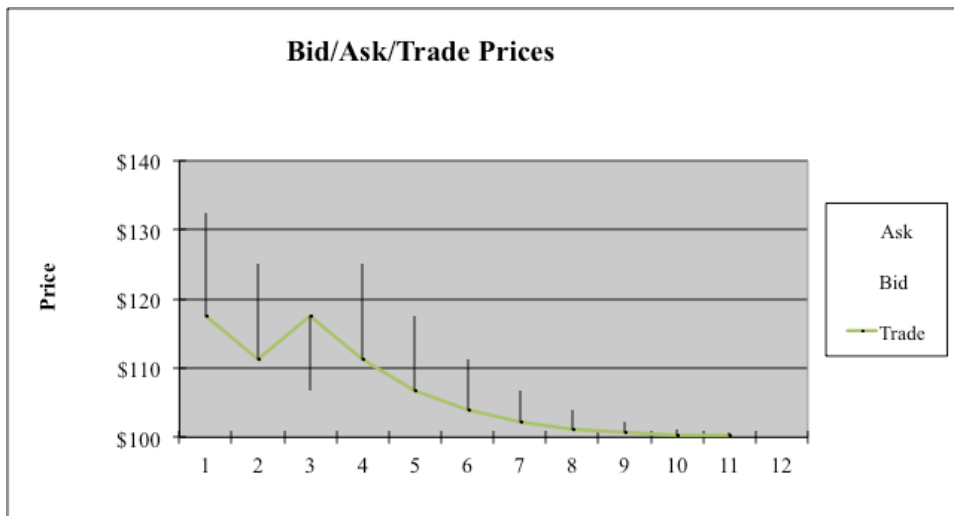
Dynamics

Tenth period: sell



Dynamics

Eleventh period: sell



Dynamics

Twelfth period: sell

