Exercise 6.1

This exercise considers a two-period model of an LOB.

- **Security**: Value $v \sim g(v) = \frac{1}{2\sigma} e^{-\frac{|v-\mu|}{\sigma}}$
  - Useful fact: $\mathbb{E}[v|v \geq z] = z + \sigma$.
- **Trader**:
  - Type I. With prob. π_0, knows v and maximizes exp. utility
  - Type N. With prob. $1 - \pi_0$, buys/sells (with equal prob.) x_N shares, with $x_N \sim f(x_N) = \frac{1}{2}\theta e^{-\theta|x_N|}$
- **Tick**: Tick size is zero, prices are continuous

Exercise 6.1.a

(a) Let $q(A)$ be the cumulative depth up to ask price A in the book and A^* be the lowest ask price in the LOB. Show that when $v \geq A^*$, the optimal strategy $x^*(v)$ of the informed trader is to buy $x^*(v) = q(v)$ shares.

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- Let $p(q) \equiv \{A : x = q(A)\}$ be the marginal price schedule.

  Note that it's just the inverse of $q(A)$: $p(q(A)) = A$ and $q(p(q')) = q'$.
- The trader's profit from buying x units is $\int_0^x (v - p(q))\, dq$.
- Take the FOC wrt. x

$$v - p(x^*) = 0 \Leftrightarrow p(x^*) = v.$$

- Denote the optimal strategy by $x^*(v)$. Then: $q(p(x^*(v))) = x^*(v) = q(v)$.

Exercise 6.1.b

(b) Using part (a) and the zero-profit condition show that in eqm:

$$q(A) = \frac{1}{\theta}\left[\ln\left(\frac{1-\pi_0}{\pi_0}\right) + \ln\left(\frac{A-\mu}{\sigma}\right) + \frac{A-\mu}{\sigma}\right] \text{ if } A > A^*.$$

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Zero-profit condition is $p(q) = \mathbb{E}[v|x \geq q]$ (x is the incoming mkt order), can be expanded into:

$$p(q) = \pi(q)\mathbb{E}[v|v \geq p(q)] + (1-\pi(q))\mathbb{E}[v],$$

where the first term incorporates the $v = p(x^*(v))$ shown in (a), and

$$\begin{aligned}\pi(q) \equiv \mathbb{P}(I|x \geq q) &= \frac{\mathbb{P}(I)\mathbb{P}(x \geq q|I)}{\mathbb{P}(x \geq q)} \\ &= \frac{\pi_0\mathbb{P}(v \geq p(q))}{\pi_0\mathbb{P}(v \geq p(q)) + (1-\pi_0)\mathbb{P}(x_N \geq q)}.\end{aligned}$$

Exercise 6.1.b (2)

- Compute the ingredients of $\pi(q)$:

$$\mathbb{P}(v \geq p(q)) = \int_{p(q)}^{\infty} g(v)dv = \frac{1}{2}e^{-\frac{p(q)-\mu}{\sigma}}$$

$$\mathbb{P}(x_N \geq q) = \int_{q}^{\infty} f(x_N)dx_N = \frac{1}{2}e^{-\theta q}$$

- Plug them back into $\pi(q)$:

$$\begin{aligned}\pi(q) &= \frac{\pi_0 \frac{1}{2} e^{-\frac{p(q)-\mu}{\sigma}}}{\pi_0 \frac{1}{2} e^{-\frac{p(q)-\mu}{\sigma}} + (1-\pi_0)\frac{1}{2}e^{-\theta q}} \\ &= \frac{e^{\theta q-\frac{p(q)-\mu}{\sigma}}}{e^{\theta q-\frac{p(q)-\mu}{\sigma}} + \frac{1-\pi_0}{\pi_0}}\end{aligned}$$

Exercise 6.1.b (3)

- Going back to the marginal price, we had

$$p(q) = \pi(q)\mathbb{E}[v|v \geq p(q)] + (1-\pi(q))\mathbb{E}[v]$$

- Recall the hint: $\mathbb{E}[v|v \geq p(q)] = p(q) + \sigma$. Plug this and $\pi(q)$ into the expression above:

$$\begin{aligned} & p(q) = \pi(q)(p(q)+\sigma) + (1-\pi(q))\mu \\ \Leftrightarrow\ & \frac{1-\pi_0}{\pi_0} \cdot \frac{p(q)-\mu}{\sigma} = e^{\theta q - \frac{p(q)-\mu}{\sigma}} \\ \Leftrightarrow\ & \ln\left(\frac{1-\pi_0}{\pi_0}\right) + \ln\left(\frac{p(q)-\mu}{\sigma}\right) = \theta q - \frac{p(q)-\mu}{\sigma} \end{aligned}$$

Finally, solve for q (recall $p(q(A)) = A$) to get the result.

Exercise 6.1.c

(c) Show that the book becomes thinner on the ask side when (i) π_0 increases or (ii) σ increases. What is the economic intuition for this result.

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- **Share of informed traders**.

$$\frac{\partial q(A)}{\partial \pi_0} = \frac{1}{\theta}\left[-\frac{\pi_0}{1-\pi_0}\cdot\frac{1}{\pi_0^2}\right] = \frac{1}{\theta}\left[-\frac{1}{\pi_0(1-\pi_0)}\right] < 0$$

More informed trading implies greater risk when posting orders

Exercise 6.1.c

(c) Show that the book becomes thinner on the ask side when (i) π_0 increases or (ii) σ increases. What is the economic intuition for this result.

$$q(A) = \frac{1}{\theta}\left[\ln\left(\frac{1-\pi_0}{\pi_0}\right) + \ln\left(\frac{A-\mu}{\sigma}\right) + \frac{A-\mu}{\sigma}\right] \text{ if } A > A^*.$$

- **Value uncertainty**.

$$\frac{\partial q(A)}{\partial \sigma} = \frac{1}{\theta}\left[-\frac{\sigma}{A-\mu}\frac{A-\mu}{\sigma^2} - \frac{A-\mu}{\sigma^2}\right] = \frac{1}{\theta}\left[-\frac{1}{\sigma} - \frac{A-\mu}{\sigma^2}\right] < 0$$

More value uncertainty implies greater risk when posting orders