Final exam solutions

Problem 1: Estimating spread with informative order flow

Suppose you want to estimate the average spread for a given stock, but only have access to trade data (prices, directions, volumes). You hence decide to use the Roll's model that we have seen in class. At the same time, you find its assumption about the uninformative order flow unrealistic and want to relax it. You believe instead that some traders are informed, and the prices reflect that (as in, e.g., the Glosten-Milgrom world), which leads to $\mathbb{E}(d_t \epsilon_t) = \rho > 0$ in the Roll's model.

1. Can you estimate ρ from trade data? If yes, explain in a few words how you would do it (you do not need to spell out the formal estimation procedure). If no, explain why and suggest another way to estimate spread.
2. Assuming you know (or have a good estimate for) the value of ρ, provide an estimator for spread S.

Solution

Question 1. By definition, $\rho \equiv \mathbb{E}(d_t \epsilon_t)$. It is said in the problem that we can observe trade directions d_t. However, $\epsilon_t = m_t - m_{t-1}$ is not directly observed, since midquotes m_t are unknown. It is, however, true, that $m_t = p_t - \frac{d_t S}{2}$, so $\epsilon_t = \Delta p_t - \frac{\Delta d_t S}{2}$ – it is given by the innovation in price, adjusted for the spread. We could then estimate ρ simultaneously with S by using, e.g., the Generalized Method of Moments, given the estimator for S presented below and

$$\hat{\rho} = \mathbb{E}\left[d_t\left(\Delta p_t - \frac{\Delta d_t S}{2}\right)\right].$$

In principle, since we are interested in ϵ_t as an informational innovation, we could also look at a long-run price impact $p_{t+\tau} - p_t$ for some large enough τ instead of Δp_t, adjusted for the spread accordingly. This would filter out any short-run fluctuations caused by inventory risk or order processing costs.

Question 2. Suppose the following:

1. All trades have the same size. $d = 1$: buy, $d = -1$: sell
2. Arriving orders are i.i.d. with $\mathbb{P}(d_t = 1) = \frac{1}{2}$
3. Midquote is random walk: $m_t = m_{t-1} + \epsilon_t$, where ϵ_t are i.i.d. shocks
4. Market orders are informative about price movements: $\mathbb{E}(d_t \epsilon_t) = \rho > 0$ (but maintain $\mathbb{E}(d_t \epsilon_{t+1}) = 0$).
5. Spread $S = a_t - b_t$ is constant.

Then

$$p_t = m_t + \frac{d_t S}{2}.$$

$$
\begin{aligned}
Cov(\Delta p_t, \Delta p_{t-1}) &= Cov(p_t - p_{t-1}, p_{t-1} - p_{t-2}) \\
&= Cov\left(\frac{S}{2}d_t - \frac{S}{2}d_{t-1} + \epsilon_t, \frac{S}{2}d_{t-1} - \frac{S}{2}d_{t-2} + \epsilon_{t-1}\right) \\
&= \frac{S^2}{4}Cov\left(d_t - d_{t-1}, d_{t-1} - d_{t-2} + \frac{2}{S}\epsilon_{t-1}\right) \\
&= \frac{S^2}{4}\mathbb{E}\left[(d_t - d_{t-1})\left(d_{t-1} - d_{t-2} + \frac{2}{S}\epsilon_{t-1}\right)\right] \\
&= \frac{S^2}{4}\mathbb{E}\left[-d_{t-1}^2 - \frac{2}{S}\rho\right] = \frac{-S^2 - 2\rho S}{4}
\end{aligned}
$$

Denoting $C \equiv Cov(\Delta p_t, \Delta p_{t-1})$, we then have

$$
\begin{aligned}
S^2 + 2\rho S + 4C = 0 \\
\Rightarrow S = -\rho \pm \sqrt{\rho^2 - 4C}.
\end{aligned}
$$

Recall that for $\rho = 0$, our estimator was $\hat{S} = 2\sqrt{-Cov(\Delta p_t, \Delta p_{t-1})}$. This corresponds to the positive root above, hence our new spread estimator is

$$\hat{S} = -\rho + \sqrt{\rho^2 - 4Cov(\Delta p_t, \Delta p_{t-1})}$$

Note: the book talks briefly about the Roll model with adverse selection in Box 5.1. It argues that in the presence of adverse selection, the Roll estimator underestimates the spread by a factor of $\sqrt{(\lambda + \gamma)/\gamma}$. In order for this answer to be accepted, the student must then explain what λ and γ are, and how they can be estimated.

Problem 2: Order splitting

Splitting a large order into multiple smaller orders is an important instrument that traders have for moderating their price impact, but it also has general equilibrium implications for order pricing. This problem explores these implications in the context of the Glosten-Milgrom model.

There is one asset traded in a dealer market with a representative (competitive) risk-neutral dealer, who is tasked with quoting an ask price a_1 and a bid price b_1 for one unit of the asset at all times. The dealer believes that the asset's fundamental value is distributed as $v \sim U[\mu - \sigma, \mu + \sigma]$.

The dealer's belief is that any arriving trader is risk-neutral and either informed with probability π, or uninformed with probability $1 - \pi$. An informed trader knows v and selects the profit-maximizing trade given v. An uninformed (noise) trader wants to buy or sell with equal probabilities, and would like to trade either one (with probability $1 - \gamma$) or two (w.p. γ) units of the asset. To trade two units, the trader remains in the market after the first trade and submits another order. The dealer observes the trader's identity and can offer different quotes a_2 and b_2 to a repeat trader.

1. Explain why in such a market, an informed trader would never profit by trading more than two units.
2. Calculate the ask price a_1 that a dealer would quote for the first unit.
3. Calculate the ask price for "repeat" purchases, a_2.
4. Compare these quotes with what would have been offered in a pure limit order book market (without a dealer), according to the Glosten model (with no order display cost). Would the market-order-traders prefer to trade in a dealer market or a LOB market?
5. Now assume instead that the dealer cannot see the trader's identity, so quote a_2 is also "public". This

means that it is taken either by the original trader who wanted to trade two units, or by a second trader willing to buy (the dealer's beliefs about the second trader are the same as about the first). Calculate the ask quote a_2^O. How does it compare to the quote a_2 derived above? Would the traders prefer their identity to be observable or not?

Solution

1. An uninformed trader never trades more than two units. Hence if the dealer observes that a trader wants to trade a third unit, the immediate inference is that the trader must be informed. The respective quotes a_3, b_3 are then determined from the zero-profit equations $a_3 = \mathbb{E}[v \mid v \geq a_3]$, $b_3 = \mathbb{E}[v \mid v \leq b_3]$ (which already include the informed trader's optimal strategy "buy only if $v \geq a_3$; sell only if $v \leq b_3$"). These have unique solutions $a_3 = \mu + \sigma$, $b_3 = \mu - \sigma$. Trading at these prices yields zero profit to an informed trader.

   Note that the same logic applies to switching trade directions (submitting a "buy" after a "sell" or vice versa).

2. Let $x \in \{-2, -1, 1, 2\}$ denote the total signed quantity that the trader plans to buy. The informed trader buys the first unit ($x \geq 1$) if the marginal profit from doing so is positive, i.e., if $v - a_1 \geq 0 \iff v \geq a_1$. It is convenient to introduce $\sigma_1 = a_1 - \mu$ and say that the informed trader buys the first unit at price a_1 if $v > \mu + \sigma_1$.

   The zero-profit condition for the dealer, $a_1 = \mathbb{E}[v \mid x \geq 1]$, can be expanded using the law of iterated expectations as follows, where I and U denote the events of trader being informed or uninformed, respectively.

$$\begin{aligned} a_1 &= \mathbb{E}[v \mid x \geq 1] \\ &= \mathbb{E}[v \mid U, x \geq 1] \cdot \mathbb{P}[U \mid x \geq 1] + \mathbb{E}[v \mid I, x \geq 1] \cdot \mathbb{P}[I \mid x \geq 1] \\ &= \mu \cdot \frac{(1-\pi)/2}{\pi \frac{\sigma - \sigma_1}{2\sigma} + \frac{1-\pi}{2}} + \left(\frac{(\mu + \sigma) + (\mu + \sigma_1)}{2} \right) \cdot \frac{\pi \frac{\sigma - \sigma_1}{2\sigma}}{\pi \frac{\sigma - \sigma_1}{2\sigma} + \frac{1-\pi}{2}} \\ &= \mu + \frac{\sigma + \sigma_1}{2} \cdot \frac{\pi \frac{\sigma - \sigma_1}{2\sigma}}{\pi \frac{\sigma - \sigma_1}{2\sigma} + \frac{1-\pi}{2}}. \end{aligned}$$

   From the trader's optimality above, $a_1 = \mu + \sigma_1$, which yields:

$$\begin{aligned} \mu + \sigma_1 &= \mu + \frac{\sigma + \sigma_1}{2} \cdot \frac{\pi \frac{\sigma - \sigma_1}{2\sigma}}{\pi \frac{\sigma - \sigma_1}{2\sigma} + \frac{1-\pi}{2}} \\ \Leftrightarrow \frac{\sigma_1}{2} \cdot \left(2 - \frac{\pi \frac{\sigma - \sigma_1}{2\sigma}}{\pi \frac{\sigma - \sigma_1}{2\sigma} + \frac{1-\pi}{2}} \right) &= \frac{\sigma}{2} \cdot \frac{\pi \frac{\sigma - \sigma_1}{2\sigma}}{\pi \frac{\sigma - \sigma_1}{2\sigma} + \frac{1-\pi}{2}} \\ \Leftrightarrow 2\sigma\sigma_1 - \pi\sigma_1^2 &= \pi\sigma^2 \\ \Rightarrow \sigma_1 &= \frac{1 - \sqrt{1 - \pi^2}}{\pi} \sigma \end{aligned}$$

   (since the positive root yields $\sigma_1 > \sigma$, which is nonsense). Hence we have $a_1 = \mu + \sigma_1$, where σ_1 is as above

3. Similarly to the above, the informed trader buys the second unit at price a_2 only if $v > \mu + \sigma_2$, where

$\sigma_2 = a_2 - \mu$. We are looking for $\sigma_2 \in (\sigma_1, \sigma)$. The dealer's zero-profit condition now is

$$\begin{aligned} a_2 &= \mathbb{E}[v \mid x \geq 2] \\ &= \mathbb{E}[v \mid U, x \geq 2] \cdot \mathbb{P}[U \mid x \geq 2] + \mathbb{E}[v \mid I, x \geq 2] \cdot \mathbb{P}[I \mid x \geq 2] \\ &= \mu \cdot \frac{(1-\pi)\frac{\gamma}{2}}{\pi\frac{\sigma-\sigma_2}{2\sigma} + (1-\pi)\frac{\gamma}{2}} + \left(\frac{(\mu+\sigma)+(\mu+\sigma_2)}{2}\right) \cdot \frac{\pi\frac{\sigma-\sigma_2}{2\sigma}}{\pi\frac{\sigma-\sigma_2}{2\sigma} + (1-\pi)\frac{\gamma}{2}} \\ &= \mu + \frac{\sigma+\sigma_2}{2} \cdot \frac{\pi\frac{\sigma-\sigma_2}{2\sigma}}{\pi\frac{\sigma-\sigma_2}{2\sigma} + (1-\pi)\frac{\gamma}{2}}. \end{aligned}$$

Plugging in $a_2 = \mu + \sigma_2$, we get

$$\begin{aligned} \sigma_2 &= \frac{\sigma+\sigma_2}{2} \cdot \frac{\pi\frac{\sigma-\sigma_2}{2\sigma}}{\pi\frac{\sigma-\sigma_2}{2\sigma} + (1-\pi)\frac{\gamma}{2}} \\ \Leftrightarrow 0 &= \pi\sigma^2 - 2\sigma\left(\pi + (1-\pi)\gamma\right)\sigma_2 + \pi\sigma_2^2 \\ \Rightarrow \sigma_2 &= \frac{\pi + (1-\pi)\gamma - \sqrt{(2\pi + (1-\pi)\gamma)(1-\pi)\gamma}}{\pi}\sigma. \end{aligned}$$

4. Recall that in the Glosten model, a trader who submits a limit order can only condition their price on the event that their limit order executes, but not on the total order size. In our setup, there would be two limit orders on the ask side of the market for one unit each. The price of the first unit would then be $a_1 = \mathbb{E}[v \mid x \geq 1]$, and the price of the second unit $a_2 = \mathbb{E}[v \mid x \geq 2]$ – exactly the same as in the dealer model above. This serves to show that when market-order traders split their orders, the dealer has no informational advantage over the LOB traders. The market-order traders then do not care whether they trade against a dealer or a limit order book.

5. Intuitively, suppose the dealer believes that with probability λ they trade against the second order of the first trader, and with probability $1-\lambda$ against the first order of the second trader. The competitive ask quote would then satisfy $a_2^O = \lambda\mathbb{E}[v \mid x \geq 2] + (1-\lambda)\mathbb{E}[v \mid x \geq 1]$, so it would be $a_2^O \in (a_1, a_2)$.[1] Since $a_1 < a_2^O < a_2$, all market-order traders (who trade two units) would prefer an opaque market, since they get better prices in such a market, due to the dealer allowing for a possibility that a new uninformed trader arrives. Note, however, that this is only true for the "first" trader in the market – it is not obvious how the prices for the second and later traders would change.

   A formal derivation is a_2^O is quite cumbersome, and the students were not expected to complete it; an intuitive argument like the one above would suffice. For completeness, the full derivation is presented below. We are looking for an ask price a_2^O (which is only relevant if the second order is a buy) in an opaque market conditional on the first order being a buy. Define again $\sigma_2^O \equiv a_2^O - \mu$ and guess that $\sigma_2^O > \sigma_1$ (verify at the end). The contingencies that generate a (buy, buy) order sequence, together with their ex ante (unconditional) probabilities, are as follows:

   (a) The first trader was uninformed and wanted to buy two units: probability $(1-\pi)\frac{\gamma}{2}$; $\mathbb{E}[v] = \mu$;[2]

   (b) The first trader was informed and v is high enough to justify two buys: probability $\pi\frac{\sigma-\sigma_2^O}{2\sigma}$; infer that $v \in [\mu+\sigma_2^O, \mu+\sigma]$, so $\mathbb{E}[v] = \mu + \frac{\sigma_2^O+\sigma}{2}$ in this case;

   (c) The first trader left after buying one unit, and the second trader wants to buy at least one unit. The first trader leaves if:

      i. uninformed, only wanted to buy one unit (probability $(1-\pi)\frac{1-\gamma}{2}$); in this case the expected

[1] This is not strictly correct, since $\mathbb{E}[v \mid x \geq 2]$ itself depends on a_2.

[2] To be clear, notation is used loosely in this list, and the expectations $\mathbb{E}[v]$ actually mean conditional expectations, conditional on the respective cases.

value of the asset conditional on the second order being a buy order is exactly $\mathbb{E}[v] = a_1$;

ii. informed, and v is not high enough to warrant buying second unit (probability $\pi\frac{\sigma_2^O-\sigma_1}{2\sigma}$; infer that $v \in [\mu+\sigma_1, \mu+\sigma_2^O]$). Note that in this case if the second trader is also informed, he would also not want to buy at price a_2^O. Hence a second buy order can only come from an uninformed trader (probability $\frac{1-\pi}{2}$ on top of the probability above) and would not provide any information about v beyond $v \in [\mu+\sigma_1, \mu+\sigma_2^O]$, hence in this case $\mathbb{E}[v] = \mu + \frac{\sigma_1+\sigma_2^O}{2}$.

The total probability of all contingencies above is

$$(1-\pi)\frac{\gamma}{2} + \pi\frac{\sigma-\sigma_2^O}{2\sigma} + (1-\pi)\frac{1-\gamma}{2} + \pi\frac{\sigma_2^O-\sigma_1}{2\sigma}\cdot\frac{1-\pi}{2} = \frac{1-\pi}{2}\cdot\left(1+\pi\frac{\sigma_2^O-\sigma_1}{2\sigma}\right) + \pi\frac{\sigma-\sigma_2^O}{2\sigma}$$

Taking the expectation of v over the cases above, we get that

$$\begin{aligned}
a_2^O = \mu + \sigma_2^O = & \frac{(1-\pi)\frac{\gamma}{2}}{\frac{1-\pi}{2}\cdot\left(1+\pi\frac{\sigma_2^O-\sigma_1}{2\sigma}\right)+\pi\frac{\sigma-\sigma_2^O}{2\sigma}}\cdot\mu + \frac{\pi\frac{\sigma-\sigma_2^O}{2\sigma}}{\frac{1-\pi}{2}\cdot\left(1+\pi\frac{\sigma_2^O-\sigma_1}{2\sigma}\right)+\pi\frac{\sigma-\sigma_2^O}{2\sigma}}\cdot\left(\mu+\frac{\sigma_2^O+\sigma}{2}\right) \\
& + \frac{(1-\pi)\frac{1-\gamma}{2}}{\frac{1-\pi}{2}\cdot\left(1+\pi\frac{\sigma_2^O-\sigma_1}{2\sigma}\right)+\pi\frac{\sigma-\sigma_2^O}{2\sigma}}\cdot a_1 + \frac{\pi\frac{\sigma_2^O-\sigma_1}{2\sigma}\cdot\frac{1-\pi}{2}}{\frac{1-\pi}{2}\cdot\left(1+\pi\frac{\sigma_2^O-\sigma_1}{2\sigma}\right)+\pi\frac{\sigma-\sigma_2^O}{2\sigma}}\cdot\left(\mu+\frac{\sigma_1+\sigma_2^O}{2}\right) \\
= & \mu + \frac{\pi\frac{\sigma-\sigma_2^O}{2\sigma}\cdot\frac{\sigma_2^O+\sigma}{2} + (1-\pi)\frac{1-\gamma}{2}\cdot\sigma_1 + \pi\frac{\sigma_2^O-\sigma_1}{2\sigma}\cdot\frac{1-\pi}{2}\cdot\frac{\sigma_1+\sigma_2^O}{2}}{\frac{1-\pi}{2}\cdot\left(1+\pi\frac{\sigma_2^O-\sigma_1}{2\sigma}\right)+\pi\frac{\sigma-\sigma_2^O}{2\sigma}}
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow \sigma_2^O\cdot\left[\frac{1-\pi}{2}\cdot\left(1+\pi\frac{\sigma_2^O-\sigma_1}{2\sigma}\right)+\pi\frac{\sigma-\sigma_2^O}{2\sigma}\right] = \frac{1-\pi}{2}(1-\gamma)\sigma_1 + \frac{\pi}{4\sigma}\left(\sigma^2 - \frac{1+\pi}{2}(\sigma_2^O)^2 - \frac{1-\pi}{2}\sigma_1^2\right) \\
&\iff (\sigma_2^O)^2\cdot\frac{\pi(1-\pi)}{4\sigma} + \sigma_2^O\cdot\left(\frac{\pi(1-\pi)\sigma_1}{2\sigma}-1\right) + \left((1-\pi)(1-\gamma)\sigma_1 + \frac{\pi\sigma}{2} - \frac{\pi(1-\pi)\sigma_1^2}{4\sigma}\right) = 0.
\end{aligned}$$

The above is a quadratic equation w.r.t. σ_2^O, hence can be solved using the standard methods. Plotting the solution numerically shows that indeed $\sigma_2^O < \sigma_2$ (and demonstrates which root is relevant). For low enough π and γ it is also true that $\sigma_2^O > \sigma_1$, as assumed. If the parameters are such that this does not hold, then $a_2^O = a_1$ is the corner solution.

Problem 3: The fall of FTX

FTX, a large crypto exchange, has collapsed in the Fall of 2022. You can find the timeline of the collapse in the article from Investopedia attached at the end of this exam.[3] You are to answer the following question:

"How does the fall of FTX affect the other participants of the crypto market?"

1. Consult a chatbot/AI/LLM (hereinafter referred to as LLM) on this issue.[4] Show both your prompt and the response (and mention which LLM you used).

   NOTE: for best results, you may want to phrase your own prompt, rather than just copy the question above.

2. Discuss the response you got. Does it make sense? Is it applicable to the situation at hand?

3. Is there anything you would like to add to the LLM response, using the knowledge you obtained in the

[3] Original article available at `https://www.investopedia.com/what-went-wrong-with-ftx-6828447`.

[4] For example, you can use openAI's ChatGPT: `https://chat.openai.com`.

course?

Solution

1. The student is evaluated based on the quality of their prompt. E.g., it should be immediate that the LLM did not take this course and likely has no idea it exists, hence mentioning the course is not constructive. At the same time, mentioning "microstructure" may in principle frame the LLM's responses in a more relevant key. Finally, given that the collapse of FTX is a recent event, the LLMs are unlikely to have information relevant to the specific case (which, e.g., openAI usually mentions in its responses) – so it makes sense to ask more generic, counterfactual questions, such as "How would a collapse of a large cryptocurrency exchange affect other market participants?" Finally, the investopedia text is presented in the exam to give the unaware students a chance to get acquainted with the case, not to be fed to the LLM. It is implied that the text is short enough to be processed by the student without the help of the LLM – and the benefit of the LLM is in accessing the broader body of knowledge, as opposed to extracting core points from a short text.
2. In my experience, the LLM responses are typically vague, but mostly relevant. If forced to speculate on the specific case of FTX, the LLM can hypothesize about the reasons for the collapse that do not have much in common with the realized timeline (such as security issues); these can be rebuked by the student. Alternatively, if the LLM suggests hypothetical consequences that would be instantly testable using widely accessible data, the student is expected to do the fact-checking (e.g., "investors can lose confidence in crypto" or "other exchanges would gain market share by capturing orphaned traders" can both be easily tested by looking up trading volumes on competing crypto exchanges, such as Coinbase or Binance).
3. Open question. An easy answer would be to discuss the consequences of market fragmentation, which the LLMs tend to not mention (fragmentation might either increase or decrease after FTX demise, depending on where traders migrate to), and with it market transparency (the ease to aggregate information from different platforms).