Financial Markets Microstructure: Problem Set 1

Egor Starkov

Københavns Universitet, Spring 2023

Problem 1

Exercise 4 in chapter 3 (pp. 125-126):

Consider the one-period Glosten-Milgrom model with $v \in \{v^L, v^H\}$ and $\mathbb{P}(v^H) = 1/2$. The dealer is competitive and risk neutral, so prices are equal to expected values. With probability $1 - \pi$ the trader is a noise trader who buys and sells with equal probability, with probability π he is a potential insider. The potential insider can, before submitting a trade, privately acquire information that perfectly reveals v at some fixed cost $c > 0$. If no information is acquired, the potential insider has no private information about v. As usual, suppose the dealer cannot distinguish the traders.

(a). Suppose an insider decided to not acquire any information. What trading strategy maximizes his expected profit?

> **Solution:** Since $a > \mu > b$ in equilibrium (you can verify this by solving part (b) for arbitarry trading strategy of the uninformed insider), a profit-maximizing uninformed insider would choose to abstain from trading, since it yields a payoff of zero, and expected payoff from any trade in the absence of information is negative.

(b). Compute the bid and ask prices set by the dealer, assuming that he believes that the insider acquires information with probability $\varphi \in [0, 1]$.

> **Solution:** A potential insider who does not acquire information will always abstain (as long as there is a positive bid-ask spread) whereas an insider who observes v^H (v^L) will buy (sell). (You can verify this in the end.)

Use Bayes' Rule to calculate the probability that the trader is an *insider* given a buy order is

$$\mathbb{P}(I|\text{buy}) = \frac{\mathbb{P}(I) \cdot \mathbb{P}(\text{buy}|I)}{\mathbb{P}(\text{buy})} = \frac{\pi\varphi\frac{1}{2}}{\pi\varphi\frac{1}{2} + (1-\pi)\frac{1}{2}} = \frac{\pi\varphi}{\pi\varphi + (1-\pi)}.$$

Let $\alpha_\varphi = \frac{\pi\varphi}{\pi\varphi+(1-\pi)}$. Similarly, we can deduce that $\mathbb{P}(I|\text{sell}) = \alpha_\varphi$. Letting $m = \frac{v^H+v^L}{2}$, the ask (a_φ) and bid (b_φ) prices are

$$\begin{aligned} a_\varphi &= \alpha_\varphi v^H + (1-\alpha_\varphi)m, \\ b_\varphi &= \alpha_\varphi v^L + (1-\alpha_\varphi)m. \end{aligned} \tag{1}$$

That is, the ask price is the probability that the trader is an (informed) insider times the value given that he is informed, plus the probability that the trader is a noise trader, times the expected value given that he is a noise trader (which is just the prior expectation m).

(c). Given these prices, determine the trading profit of an insider who has decided to acquire information.

Solution: The trader's profit from acquiring information depends on the belief of the dealer, i.e. the probability that the dealer attaches to the trader acquiring information. Suppose the dealer's belief attaches probability φ to the trader being an informed insider. A trader who acquires information then has the following expected profit:

$$u(\varphi) = \frac{1}{2}(v^H - a_\varphi) + \frac{1}{2}(b_\varphi - v^L) - c = (1-\alpha_\varphi)\frac{v^H - v^L}{2} - c.$$

Notice that the dealer's beliefs enters only through the prices.

(d). There exists $\bar{c}$ such that if $c > \bar{c}$, then the potential insider never acquires information (i.e., in equilibrium, $\varphi = 0$). Calculate $\bar{c}$.

Solution: Notice that $u'(\cdot) < 0$, i.e. the trader's potential profits are decreasing in the dealer's belief of the probability that he is informed. Thus, the best potential case for acquiring information is when $\varphi = 0$, that is, when the market does not expect the trader to be an informed insider. As a consequence, the trader will never acquire information when $u(0) < 0$, which corresponds to

$$(1-\alpha_0)\frac{v^H - v^L}{2} - c < 0 \Leftrightarrow c > \frac{v^H - v^L}{2} \equiv \bar{c}.$$

The higher the value volatility, as measure by the distance between the high and the low value, the greater the incentive to acquire information.

(e). There exists $\underline{c}$ such that if $c < \underline{c}$, then the potential insider always acquires information (i.e., in equilibrium, $\varphi = 1$). Calculate $\underline{c}$.

Solution: Similarly to the previous question, the worst potential case for acquiring information is when $\varphi = 1$. Hence, the trader will always acquire information when $u(1) > 0$, which corresponds to

$$(1-\alpha_1)\frac{v^H - v^L}{2} - c > 0 \Leftrightarrow c < (1-\pi)\frac{v^H - v^L}{2} \equiv \underline{c}.$$

(f). Suppose now that $c \in (\underline{c}, \bar{c})$. Find the mixed-strategy equilibrium in which the potential insider learns v with positive probability smaller than one. Describe the equilibrium fully.

Solution: Let the probability with which the trader acquires information (his strategy) be denoted by σ (recall that φ is the dealer's belief about σ). In equilibrium, we must have $\sigma^* = \varphi^* \in (0,1)$, i.e. the beliefs must be correct (otherwise, the dealer would change his beliefs). Since the trader is playing a strictly mixed information acquisition strategy, we know from standard game theory considerations that he must then be indifferent between acquiring and not acquiring information: $u(\varphi^*) = 0$. Thus,

$$(1-\alpha_{\varphi^*})\frac{v^H - v^L}{2} - c = 0 \Rightarrow \sigma^* = \varphi^* = \frac{1-\pi}{\pi}\left(\frac{v^H - v^L}{2c} - 1\right). \tag{2}$$

The equilibrium is thus such that the dealer believes that the potential insider acquires information with probability φ^* given by (2) and quotes bid and ask prices $a_{\varphi^*}, b_{\varphi^*}$ as given by (1); and the potential insider learns v with probability σ^* given by (2), submits a buy order if learned that $v = v^H$, a sell order if $v = v^L$, and no order if learned nothing.

Inspecting the expression, we can make the following observations about equilibrium information acquisition in the mixed-strategy equilibrium. Higher π means smaller probability of acquiring information. This is because more adverse selection leads to a greater spread, and therefore less profitable trades. Higher fundamental value volatility $(v^H - v^L)$ leads to higher probability of acquiring information. This is because information is worth more when values are volatile. Higher cost c leads to a smaller probability of acquiring information, since information is more expensive.

Problem 2

This problem considers an issue left open in Section 4.3 in the textbook, which explores inventory risk within the Kyle's model. At the outset of this section, it is assumed that there is no informed trading, i.e., $x = 0$ and $q = u$. Let us now relax this assumption and try to analyze the effects of dealer's risk aversion in Kyle's model in the presence of adverse selection.

So, consider a combination of the two versions of the Kyle model we have seen in class: we now have one informed trader, a pool of noise traders, and a representative competitive risk-averse dealer with mean-variance preferences, risk aversion ρ, initial asset holding z_0, and initial cash holding c_0. I.e., the dealer's utility from a [random] next period's wealth $w_{t+1} = z_{t+1}p_{t+1} + c_{t+1}$ is $U(w_{t+1}) = \mathbb{E}[w_{t+1}] - \frac{\rho}{2}\mathbb{V}(w_{t+1})$.

We are again looking for a linear equilibrium, where the insider submits an order $x = \beta v - x_0$, and the pricing schedule quoted by the dealer is $p = p_0 + \lambda q$. Now, there are four endogenous parameters, β, x_0, p_0, λ, with $\beta > 0$ and $\lambda > 0$.

(a). Take for granted some arbitrary parameters β, x_0. The dealer observes $q = x + u$. Use the result from Section 4.2 or lecture slides to find $\mathbb{E}[v|q]$ and $\mathbb{V}[v|q]$ as functions of β and the model parameters.

(Note that it is no longer the case that $q = \beta(v - \mu) + u$, but rather $q + x_0 - \beta\mu = \beta(v - \mu) + u$.)

Solution: The dealer wealth at the end of the trading day is

$$w = v(z_0 - q) + c_0 + pq$$

To simplify the problem, we will define a new variable that will have some desirable properties (you can solve the problem without doing this as well). Let

$$z = q + x_0 - \beta\mu. \tag{3}$$

Notice that z is normally distributed with $\mathbb{V}[z] = \mathbb{V}[q] = \beta^2\sigma_v^2 + \sigma_u^2$ and $\mathbb{C}[v, z] = \mathbb{C}[v, q] = \beta\sigma_v^2$. We can then use the lecture notes to get $\mathbb{E}[v|z] = \mu + \frac{\mathbb{C}[v,z]}{\mathbb{V}[z]}z$. Thus:

$$\mathbb{E}[v|z] = \mu + \frac{\beta\sigma_v^2}{\beta^2\sigma_v^2 + \sigma_u^2}z = \mu + \alpha z,$$

where $\alpha = \frac{\beta\sigma_v^2}{\beta^2\sigma_v^2 + \sigma_u^2}$, and

$$\mathbb{V}[v|z] = \frac{\sigma_v^2\sigma_u^2}{\beta^2\sigma_v^2 + \sigma_u^2}.$$

Substituting (3) into these two expressions, we get

$$\mathbb{E}[v|q] = (\mu + \alpha x_0 - \alpha\beta\mu) + \alpha q,$$

and

$$\mathbb{V}[v|q] = \frac{\sigma_v^2 \sigma_u^2}{\beta^2 \sigma_v^2 + \sigma_u^2}.$$

(b). The competitive dealer takes p as given. In equilibrium, $p = p_0 + \lambda q$, so the price reveals q – same logic as in Section 4.2.4. Rewrite $\mathbb{E}[v|q]$ and $\mathbb{V}[v|q]$ from (a) as functions of p instead of q, when $q = (p - p_0)/\lambda$. Let's call these functions $\mathbb{E}[v|p]$ and $\mathbb{V}[v|p]$.

Solution: Since we are conditioning on the same information, given that p reveals q but nothing else, the conditional expectation and variance do not change. We merely substitute $q = (p - p_0)/\lambda$ into the expressions from the previous question. This yields

$$\mathbb{E}[v|p] = (\mu + \alpha x_0 - \alpha\beta\mu) + \alpha \frac{p - p_0}{\lambda},$$

and

$$\mathbb{V}[v|p] = \frac{\sigma_v^2 \sigma_u^2}{\beta^2 \sigma_v^2 + \sigma_u^2}.$$

(c). Calculate the dealer's asset supply curve and show that the amount supplied at price p is

$$y(p) = z_0 + \frac{p - \mathbb{E}[v|p]}{\rho \mathbb{V}[v|p]}$$

(A reminder: To solve this problem, instead of the zero-profit condition you should invoke another feature of competitive agents, namely price-taking. Use the following algorithm to obtain the dealer's price schedule: (1) fix some arbitrary price p; (2) find supply amount $y(p)$ that maximizes the dealer's profit given this p; (3) invert the supply function $y(p)$ to obtain a pricing schedule $p(q)$.)

Solution: Conditional on p, choose supply q to maximize utility

$$\begin{aligned} U(w|p) &= \mathbb{E}[w] - \frac{\rho}{2}\mathbb{V}[w] \\ &= \mathbb{E}[v|p](z_0 - q) + c_0 + pq - \frac{\rho}{2}\mathbb{V}[v|p](z_0 - q)^2. \end{aligned}$$

Take the FOC wrt. q

$$-\mathbb{E}[v|p] + p + \rho\mathbb{V}[v|p](z_0 - q) = 0.$$

Solving this for q gives the desired result.

(d). When the market clears at price p, then $y(p) = q$. Can you see if there exists a $\lambda > 0$ and a number p_0, such that market clearing fits with our conjecture that $p = p_0 + \lambda q$? I.e. try to solve the following equation

$$(y(p) =) \quad z_0 + \frac{p - \mathbb{E}[v|p]}{\rho\mathbb{V}[v|p]} = \frac{p - p_0}{\lambda} \quad (= q)$$

w.r.t. λ and p_0. Note that they must be such that the equality holds for all p.

Solution: First, notice that

$$z_0 + \frac{p - \mathbb{E}[v|p]}{\rho\mathbb{V}[v|p]} = z_0 + \frac{p(\lambda - \alpha) + \alpha p_0 - \lambda(\mu + \alpha x_0 - \alpha\beta\mu)}{\lambda\rho\mathbb{V}[v|p]}.$$

We therefore have to solve $z_0 + \frac{p(\lambda-\alpha)+\alpha p_0 - \lambda(\mu+\alpha x_0 - \alpha\beta\mu)}{\lambda\rho\mathbb{V}[v|p]} = \frac{p-p_0}{\lambda}$, which can be rewritten as

$$\left[\frac{\lambda - \alpha}{\lambda\rho\mathbb{V}[v|p]}\right] p + \left[z_0 + \frac{\alpha p_0 - \lambda(\mu + \alpha x_0 - \alpha\beta\mu)}{\lambda\rho\mathbb{V}[v|p]}\right] = \left[\frac{1}{\lambda}\right] p + \left[-\frac{p_0}{\lambda}\right].$$

Matching first the coefficients for p, we get

$$\frac{\lambda - \alpha}{\lambda\rho\mathbb{V}[v|p]} = \frac{1}{\lambda},$$

which can be solved for

$$\lambda = \alpha + \rho\mathbb{V}[v|p]. \tag{4}$$

Matching the constants

$$z_0 + \frac{\alpha p_0 - \lambda(\mu + \alpha x_0 - \alpha\beta\mu)}{\lambda\rho\mathbb{V}[v|p]} = -\frac{p_0}{\lambda},$$

which can be solved for

$$\begin{aligned} p_0 &= \frac{\lambda}{\alpha + \rho\mathbb{V}[v|p]}((\mu + \alpha x_0 - \alpha\beta\mu) - \rho\mathbb{V}[v|p]z_0) \\ &= (\mu + \alpha x_0 - \alpha\beta\mu) - \rho\mathbb{V}[v|p]z_0. \end{aligned} \tag{5}$$

Equations (4) and (5) give us the optimal strategy of the dealer, conditional on the strategy of the traders.

(e). Go back to the informed trader's problem. Assuming that $p = p_0 + \lambda q$, argue that the insider chooses x to maximize $x(v - p_0 - \lambda x)$. Show that the relation is of the form $x = \beta v - x_0$ and try to relate parameters β, λ, x_0, p_0 to each other. Is there a solution for all four parameters? If the algebra is impossible, try a numerical solution.

Solution: The trader is a risk-neutral expected-utility maximizer. Conditional on the dealer's strategy $p = p_0 + \lambda q = p_0 + \lambda(x + u)$. Thus, he will maximize $\mathbb{E}[x(v - p)|x] = x(v - p_0 - \lambda x)$. This quadratic can be solved to give $x = (v - p_0)/(2\lambda)$, so in order for our conjecture about the strategy to be right, we must have

$$\beta = \frac{1}{2\lambda} \tag{6}$$

and

$$x_0 = \beta p_0. \tag{7}$$

Using this latter condition in (5) we get

$$p_0 = (\mu + \alpha\beta p_0 - \alpha\beta\mu) - \rho\mathbb{V}[v|p]z_0, \tag{8}$$

which is solved for

$$p_0 = \mu - \frac{\rho\mathbb{V}[v|p]z_0}{1 - \alpha\beta}. \tag{9}$$

We also have two conditions on λ, one coming from the dealer's optimality (4) and one from the trader's optimality (6). Use them together to get

$$\lambda = \alpha + \rho\mathbb{V}[v|p] = \frac{\beta\sigma_v^2 + \rho\sigma_u^2\sigma_v^2}{\beta^2\sigma_v^2 + \sigma_u^2} = \frac{2\lambda\sigma_v^2 + 4\lambda^2\rho\sigma_u^2\sigma_v^2}{\sigma_v^2 + 4\lambda^2\sigma_u^2}.$$

This is a quadratic equation in λ with a unique positive solution:

$$\lambda = \frac{\rho\sigma_v^2 + \sqrt{\rho^2\sigma_v^4 + \sigma_v^2/\sigma_u^2}}{2}. \tag{10}$$

Plugging into (6) this yields

$$\beta = \frac{1}{\rho\sigma_v^2 + \sqrt{\rho^2\sigma_v^4 + \sigma_v^2/\sigma_u^2}}. \tag{11}$$

Together, equations (7), (9), (10), and (11) pin down the four parameters.

(f). In case $\rho = 0$, the dealer is risk neutral, so we are back to the case of pure adverse selection. In this limit case, do you get back the equilibrium found in Section 4.2/in class? How do you find that changes in ρ affect market depth in equilibrium?

Solution: When $\rho = 0$, this becomes the book's solution for λ, and we also recover $p_0 = \mu$. In general, λ is increasing in risk aversion ρ.

Problem 3

One of the news articles assigned earlier in the course argued that corporate bond markets are significantly less liquid than stock markets. This is partly due to a more opaque trading tradition: instead of the dealers posting their quotes openly, the bond markets operate via Requests For Quotes (RFQ). In particular, an interested trader must contact the dealer with an RFQ, and then the dealer would respond with a quote. This process is somewhat time-consuming for traders, and thus costly. However, the costs of this opaqueness can be far greater than just these time costs.

Your goal is to build a model of search costs and to examine how they affect market outcomes: prices, spread and/or market depth, volume of trade if applicable. In particular, take any dealer model we considered (I suggest Glosten-Milgrom, but Kyle can probably work as well). Assume that instead of the dealers posting quotes openly, any arriving trader must approach dealers individually, paying cost c per dealer to learn that dealer's quotes. Solve your model as best you can and answer the following questions based on your results:

- How do prices (and thus liquidity/depth) in your model compare to the baseline model, in which dealers post quotes openly?
- How do prices depend on c?
- How do prices depend on the number of dealers in the market?
- In this situation, could a dealer profit by announcing quotes publicly?