Financial Markets Microstructure

Lecture 5

Glosten-Milgrom Model

Chapter 3.3 of FPR

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What did we do last time?

1. Argued that market thinness is not the only source of illiquidity
2. Poked holes in the Efficient Market Hypothesis
3. Defined price efficiency in many ways
4. Began talking about the GM model

Today

1. more Glosten-Milgrom!

This lecture:

1. Glosten-Milgrom (reviewed and continued)
2. GM example
3. GM: conclusions
4. extras

GM85: Overview

- Dynamic model, periods $t = 1, 2, ...;$
  (though we will be analyzing the stage game for a given period – essentially static)
- Two players in every period:
  - trader and dealer
  - dealer long-lived; trader new every period
  - trader can be informed or not
- One asset with fundamental value v (unknown), common belief $v \sim F(v)$

GM85: Model (1)

Trader: is either a speculator or a noise trader, can submit a market order $d_t \in \{1, -1\}$ to buy or sell one unit of the asset with fundamental value v (or do nothing, $d_t = 0$)

- Speculator (probability π): has private information about v.
  - We will usually assume speculator simply knows v (not much changes if he only has a noisy private signal about it).
  - Risk neutral, chooses his market order d_t to maximize expected profit $d_t \cdot (v - p_t)$:
- Noise trader (probability $1 - \pi$): no pvt info about v; trades for other reasons (hedging, liquidity).
  - We assume he follows some fixed strategy: buys with probability β_B; sells w.p. β_S; abstains w.p. $1 - \beta_B - \beta_S$
  - Important: this assumption is for simplicity only; this strategy can be perfectly rational! We just don't model what generates it.
    E.g., could say noise traders choose d_t to maximize profit $\mathbb{E}[d_t(v + y_t - p_t)]$, where y is t-trader's idiosyncratic valuation (due to risk, time, liquidity preferences...)

GM85: Model (2)

Dealer (market maker)

- Risk neutral
- Willing to trade exactly one unit (buy/sell/no trade) each period
- Sets bid and ask prices (for a single unit)
- Quote price before seing trade (limit order)
- Does not know whether trader is speculator or noise trader (but knows π)
- Expected profit from trade is $\mathbb{E}[-d_t(v - p_t)]$
- Competitive: prices=expected asset value conditional on information
- Trading is sequential: market orders served one by one

GM85: Model (3)

- **Equilibrium**:
  - An equilibrium consists of bid and ask prices and speculator's strategy
  - They must be such that: (i) prices are competitive (zero profit for MM), (ii) speculator best-responds to prices (maximizes expected gain).

Analysis. A: Market making

- Dealer quotes bid and ask prices on *one unit*
  - Can revise prices between each incoming trade
- Quoted ask price a_t only relevant if next incoming trader decides to buy
  - Dealer's payoff in this case is given by $\mathbb{E}[a_t - v|\Omega_{t-1}, Buy] = a_t - \mathbb{E}[v|\Omega_{t-1}, Buy]$
  - Same for bid b_t; payoff: $\mathbb{E}[v|\Omega_{t-1}, Sell] - b_t$
  - (Note payoffs above rely on risk-neutrality)
- Perfect competition among dealers implies zero expected profit from either trade type $\Rightarrow$ ask price and bid price are

$$a_t = \mathbb{E}[v|\Omega_{t-1}, Buy];$$
$$b_t = \mathbb{E}[v|\Omega_{t-1}, Sell].$$

- Notice that both sides of the equality depend on prices

Analysis. B: Informed trading

- Speculator knows v. Given prices a_t and b_t, the expected profits Π are:

$$\Pi(v, a_t, b_t, d_t) = \begin{cases} v - a_t & \text{if } d_t = 1; \qquad (\mathit{Buy}) \\ 0 & \text{if } d_t = 0; \qquad (\mathit{Abstain}) \\ b_t - v & \text{if } d_t = -1. \qquad (\mathit{Sell}) \end{cases}$$

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- Speculator's best response to (a_t, b_t) is: (assume $a_t \geq b_t$)
  - Buy when $v > a_t$, i.e. when v is large enough
  - Sell when $v < b_t$, i.e. when v is small enough
  - Abstain if $a_t > v > b_t$

Analysis. C: Equilibrium definition

Dealer must make zero profit (competition), traders must trade optimally. This gives us two equilibrium conditions.

- Let σ_t denote the speculator's strategy, where $\sigma_t(d_t|v)$ is the probability that the speculator places order d_t if value is v
- An equilibrium consists of prices (a_t, b_t) and strategy σ_t such that:

1. the ask and bid prices solve

$$a_t = \mathbb{E}[v|\Omega_{t-1}, Buy];$$
$$b_t = \mathbb{E}[v|\Omega_{t-1}, Sell],$$

   given σ_t

2. for each v, σ_t solves

$$\max_{\sigma_t} \{\sigma_t(1|v)[v - a_t] + \sigma_t(-1|v)[b_t - v]\},$$

   given (a_t, b_t).

Back to homework

GM Example

- **Single period**: Suppose only one period (drop t subscript, drop Ω)
- **Binary outcome**: $v \in \{0, 1\}$, equally likely ex ante: $\mathbb{P}(v = 1) = 0.5$.
- Suppose $0 < b < a < 1$ and noise trader's order obeys $\beta_B = \beta_S = 0.5$.

Questions:

1. What is the speculator's trading strategy?
2. Can you derive dealer's prices a and b, as a function of π?
   - If not, refresh your knowledge of conditional expectations and try again.
   - If you already read the solution in the book, try to replicate it without looking back at the book.
3. Are the resulting prices efficient? (Check all three forms)

Analysis. D: Solving for equilibrium (1)

- The orders reveal information about v. E.g., a buy order is submitted:
  - either by a noise trader (probability $(1-\pi)\beta_B$) – no new information, $\mu_t = \mu_{t-1} = \mathbb{E}[v|\Omega_{t-1}]$;
  - or by a speculator (probability $\pi\mathbb{P}(v \geq a_t|\Omega_{t-1})$) – then learn that $v \geq a_t$, so $\mu_t = \mathbb{E}[v|\Omega_{t-1}, v \geq a_t]$.
- Then a_t is given by (using Bayes' rule and law of total probability; N=Noise, I=Informed):

meaning that in the end, $a_t \geq \mu_{t-1}$.

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- Then a_t is given by (using Bayes' rule and law of total probability; N=Noise, I=Informed):

$$
\begin{aligned}
a_t &= \mathbb{E}[v|\Omega_{t-1}, Buy] \\
&= \mathbb{P}(N|\Omega_{t-1}, Buy) \cdot \mathbb{E}[v|\Omega_{t-1}, Buy, N] + \mathbb{P}(I|\Omega_{t-1}, Buy) \cdot \mathbb{E}[v|\Omega_{t-1}, Buy, I] \\
&= \frac{\mathbb{P}(Buy, N|\Omega_{t-1})}{\mathbb{P}(Buy|\Omega_{t-1})} \cdot \mathbb{E}[v|\Omega_{t-1}] + \frac{\mathbb{P}(Buy, I|\Omega_{t-1})}{\mathbb{P}(Buy|\Omega_{t-1})} \cdot \mathbb{E}[v|\Omega_{t-1}, v \geq a_t] \\
&= \frac{(1-\pi)\beta_B}{(1-\pi)\beta_B + \pi\mathbb{P}(v \geq a_t)} \cdot \mu_{t-1} + \frac{\pi\mathbb{P}(v \geq a_t|\Omega_{t-1})}{(1-\pi)\beta_B + \pi\mathbb{P}(v \geq a_t)} \cdot \mathbb{E}[v|\Omega_{t-1}, v \geq a_t],
\end{aligned}
$$

meaning that in the end, $a_t \geq \mu_{t-1}$.

Analysis. D: Solving for equilibrium (2)

- Similarly for sell orders:
  - sell order from a noise trader arrives with [unconditional] probability $(1-\pi)\beta_S$ – no new information, $\mu_t = \mu_{t-1}$;
  - sell order from a speculator arrives with probability $\pi\mathbb{P}(v \leq b_t|\Omega_{t-1})$ – then learn that $v \leq b_t$, so $\mu_t = \mathbb{E}[v|\Omega_{t-1}, v \leq b_t]$.
- Then b_t is given by:
  - so $b_t \leq \mu_{t-1}$, and we have confirmed that indeed $a_t \geq b_t$.

Analysis. D: Solving for equilibrium (2)

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- Then b_t is given by:

$$b_t = \mathbb{E}[v|\Omega_{t-1}, Sell]$$

$$= \frac{(1-\pi)\beta_S}{\mathbb{P}(Sell|\Omega_{t-1}, v)} \cdot \mathbb{E}[v|\Omega_{t-1}] + \frac{\pi\mathbb{P}(v \leq b_t|\Omega_{t-1})}{\mathbb{P}(Sell|\Omega_{t-1}, v)} \cdot \mathbb{E}[v|\Omega_{t-1}, v \leq b_t]$$

  - where $\mathbb{P}(Sell|\Omega_{t-1}, v) = (1-\pi)\beta_S + \pi\mathbb{P}(v \leq b_t)$,
  - so $b_t \leq \mu_{t-1}$, and we have confirmed that indeed $a_t \geq b_t$.

Analysis. E: Price efficiency

- $\mu_t \equiv \mathbb{E}[v|\Omega_t]$ is the expectation of v *after* the time-t trade order is observed. Note that in our model:

$$\mu_t = \begin{cases} a_t & \text{if buy order at } t; \\ b_t & \text{if sell order at } t. \end{cases}$$

- Meaning market price is efficient: $p_t = \mu_t$
  - in semi-strong form,
  - not in the strong form (equivalent to $p_t = v$)
- This is because dealers are competitive

This lecture:

Example (as in book)

- **Single period**: Suppose only one period (drop t subscript, drop Ω)
- **Binary outcome**: $v \in \{v^H, v^L\}$, with prior $\theta = \mathbb{P}(v^H)$
- **Prior value**: What is the prior value of the asset before trading?

$$\mu = \theta v^H + (1 - \theta) v^L.$$

Skip example

Example (2)

- How do we solve the model? Look for equilibrium with trade.
- Suppose $v^L < b < a < v^H$.
- Then speculator buys if $v = v^H$, sells if $v = v^L$.
  - That is, $\sigma(1|v^H) = 1$ and $\sigma(-1|v^L) = 1$
- The procedure is then the following
  1. Use the equilibrium conditions from before to calculate prices given the above speculator strategy
  2. Check that these prices satisfy $v^L < b < a < v^H$

Example (3)

- Let's solve for the ask price. First:

$$\mathbb{P}(Buy|v^H) = (1-\pi)\beta_B + \pi$$
$$\mathbb{P}(Buy|v^L) = (1-\pi)\beta_B$$

Then by Bayes' Rule

$$\begin{aligned}\mathbb{P}(v^H|Buy) &= \frac{\mathbb{P}(v^H)\mathbb{P}(Buy|v^H)}{\mathbb{P}(Buy)} \\ &= \frac{\theta[(1-\pi)\beta_B + \pi]}{(1-\pi)\beta_B + \pi\theta} \\ &= \theta + \frac{\theta(1-\theta)\pi}{(1-\pi)\beta_B + \pi\theta}\end{aligned}$$

Example (4)

- The ask price is the expected value, given a buy order:

$$\begin{aligned} a &= \mathbb{P}(v^H|Buy)v^H + [1 - \mathbb{P}(v^H|Buy)]v^L \\ &= \left[\theta + \frac{\theta(1-\theta)\pi}{(1-\pi)\beta_B + \pi\theta}\right] v^H + \left[1 - \left(\theta + \frac{\theta(1-\theta)\pi}{(1-\pi)\beta_B + \pi\theta}\right)\right] v^L \\ &= \mu + \frac{\theta(1-\theta)\pi}{(1-\pi)\beta_B + \pi\theta}(v^H - v^L). \end{aligned}$$

- Doing a similar exercise for b we find

$$b = \mu - \frac{\theta(1-\theta)\pi}{(1-\pi)\beta_S + \pi(1-\theta)}(v^H - v^L)$$

- Finally, we must check that our assumption holds: easy to check that $v^H > a > b > v^L$. Hence, this is an equilibrium

Example: Lessons

$$a - \mu = \frac{\theta(1-\theta)\pi}{(1-\pi)\beta_B + \pi\theta}(v^H - v^L)$$

$$\mu - b = \frac{\theta(1-\theta)\pi}{(1-\pi)\beta_S + \pi(1-\theta)}(v^H - v^L)$$

- Add the two expressions to get bid-ask spread $S = a - b$
  - S increases in π: more informed trading exacerbates adverse selection. Opposite for $(\beta_B + \beta_S)$.
  - If $\beta_B = \beta_S = 1/2$, S is increasing in $\theta(1-\theta)$, i.e. spread higher when dealer faces greater initial uncertainty about v. Same for $(v^H - v^L)$.

Example: Price discovery

- Return to the multiperiod setting. One unit traded every period, v persistent.
- Trade flow is informative – trades have long-lasting effect on prices
- Each order conveys information, dealers learn, and

$$p_t \longrightarrow v$$

Dynamics

prices strong-form efficient in the long run

Example: Price discovery

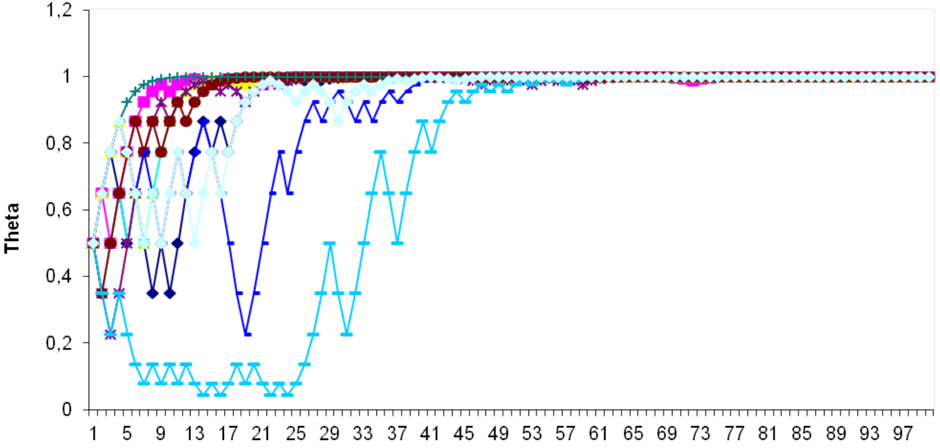
- Return to the multiperiod setting. One unit traded every period, v persistent.
- Trade flow is informative – trades have long-lasting effect on prices
- Each order conveys information, dealers learn, and

$$p_t \to v$$ Dynamics

  prices strong-form efficient in the long run
- Speed of price discovery increasing in π
  - Trade-off between price discovery and liquidity

Example: Simulation

Dealer beliefs: Each curve shows the evolution of dealer's beliefs in each run (10 runs of 100 orders)



This lecture:

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2. GM example
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4. extras

Model overview: Glosten and Milgrom

Model

- **Dealer model**: Prices are set each period, discriminative, normally competitive (zero profits)
- **Non-market clearing**: Only one unit traded - not market clearing (traders may wish to buy/sell more)
- Only **fundamental value** matters, no speculation/resale

Discussion

- **Insights**: Adverse selection as a driver of the spread
- **Shortcomings**: Trade fixed amount, trade once, no resale
- **Advantages**: (Relatively) simple analysis, flexible

Summary

What did we learn from the Glosten and Milgrom model?

1. Information, prices and the spread
   - Prices will reflect the information revealed by trades
   - The spread is increasing in informational asymmetry (adverse selection) and in uncertainty about asset value
2. Informational efficiency
   - Prices are always semi-strong efficient, in the long run also strong-form efficient
3. Noise trading
   - Noise trading keeps the market liquid and improves spreads
   - Informed speculation increases spreads, but improves price discovery - dilemma for regulators

Homework

- Reading:
  - Read two articles on absalon on how ESMA restricted trading and binary options and SEC restricted trading in certain stocks.
  - What is the difference between the underlying assets in the two cases?
  - Explain ESMA's decision using GM model.
- Solving:
  - FPR chapter 3, exercise 3 (GM model where speculators are not perfectly informed, but instead receive a signal about the value of the asset)
  - GM example with $v \sim U[0, 1]$ (rest as in the problem assigned before today; goal: derive the equilibrium bid and ask prices)

This lecture:

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Dynamics

Suppose we are in the simple binary model with the following parameters

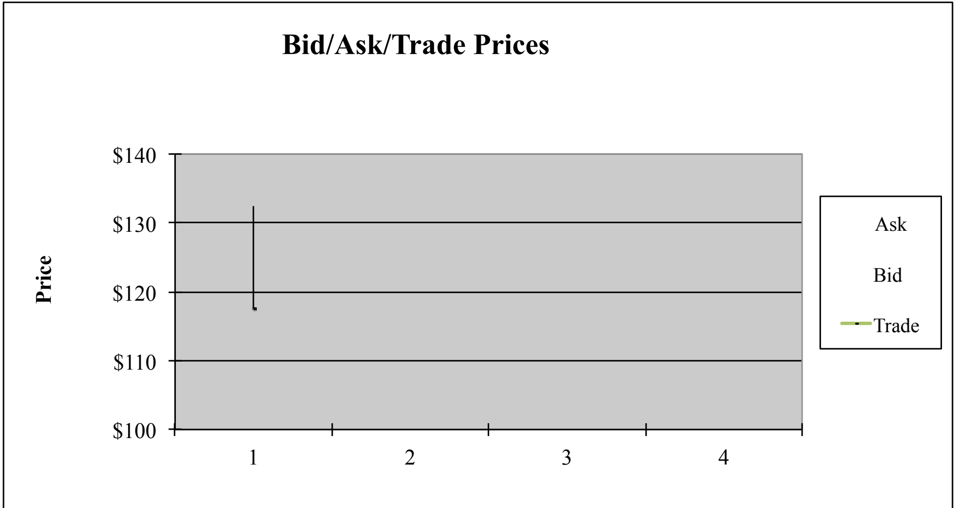
- Probability of informed speculators: $\pi = 0.3$
- Probability (ex ante) of high value: $\theta = 0.5$
- $v^H = 150$ and $v_L = 100$

Consider 12 periods, with the following sequence of buys (b) and sells (s)

$$ssbsssssssss$$

Dynamics

First period: sell



Dynamics

Second period: sell

Dynamics

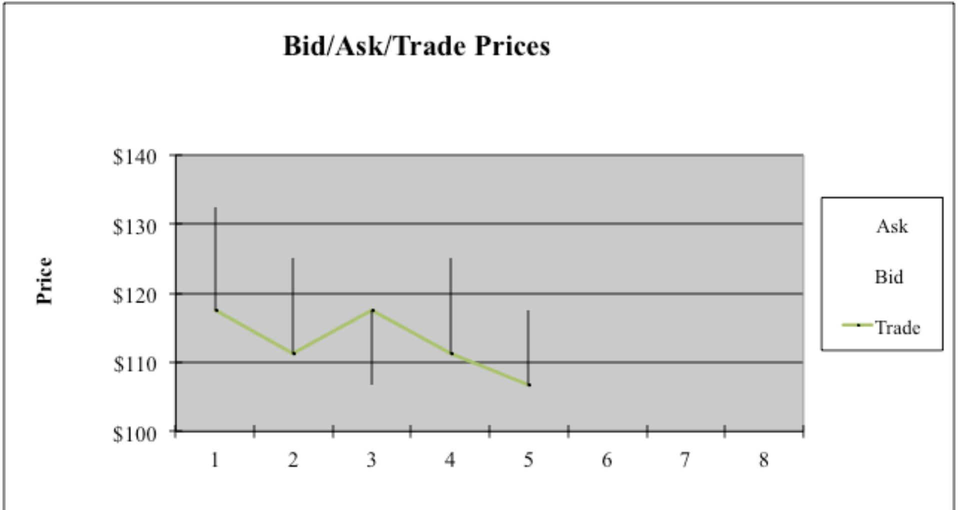
Third period: buy

Dynamics

Fourth period: sell

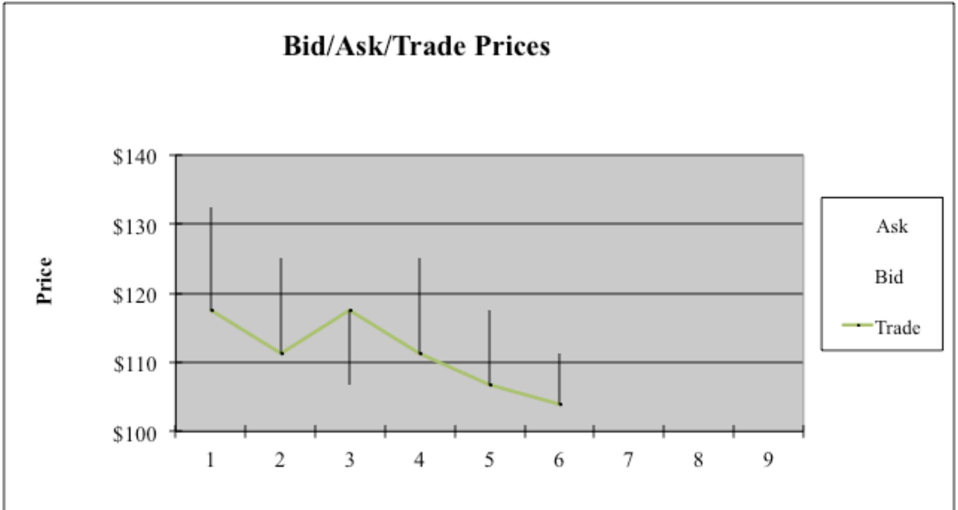
Dynamics

Fifth period: sell



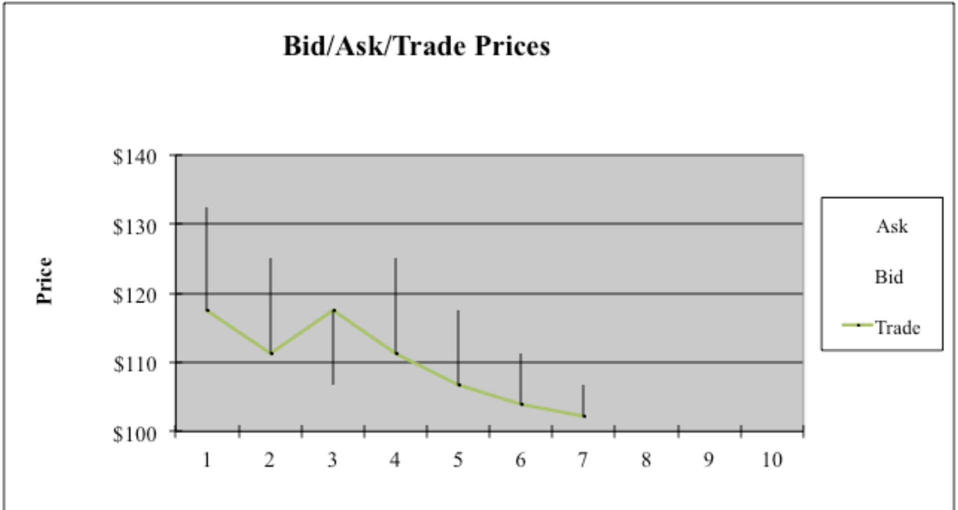
Dynamics

Sixth period: sell



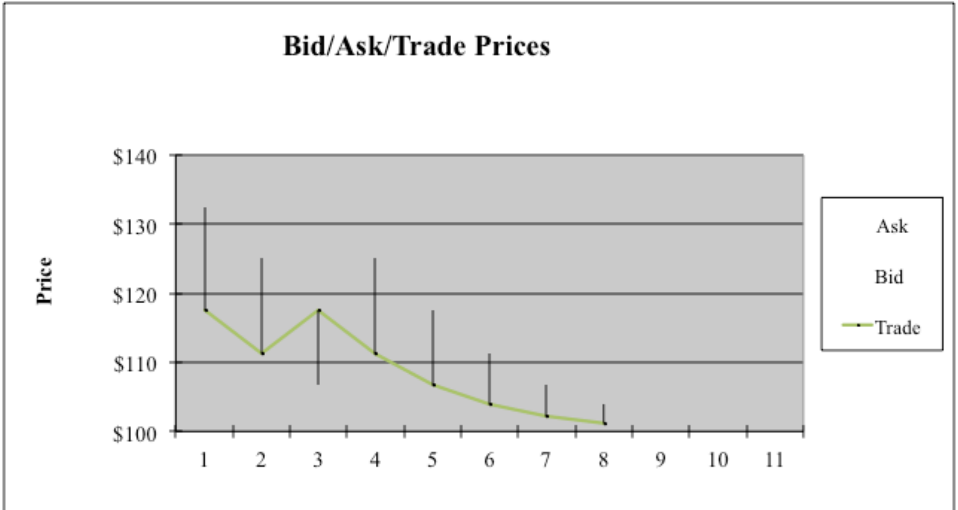
Dynamics

Seventh period: sell



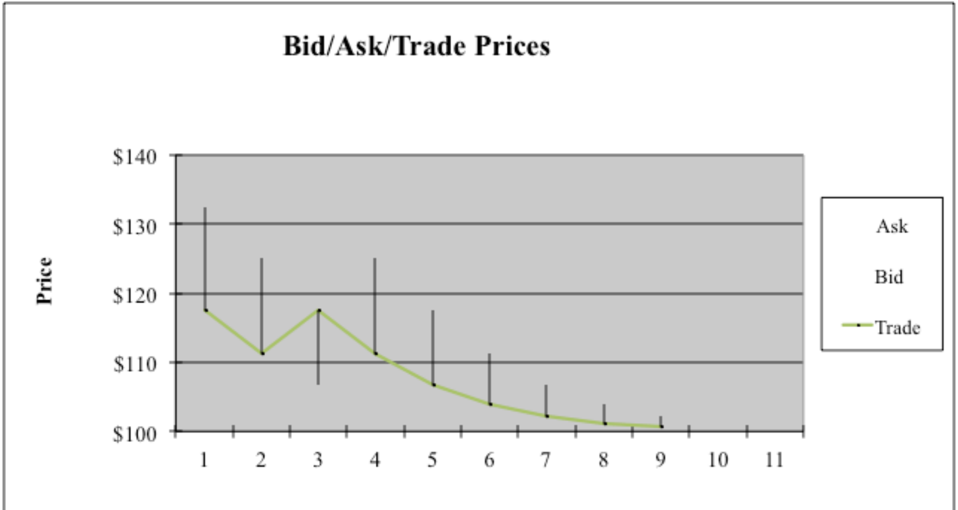
Dynamics

Eigth period: sell



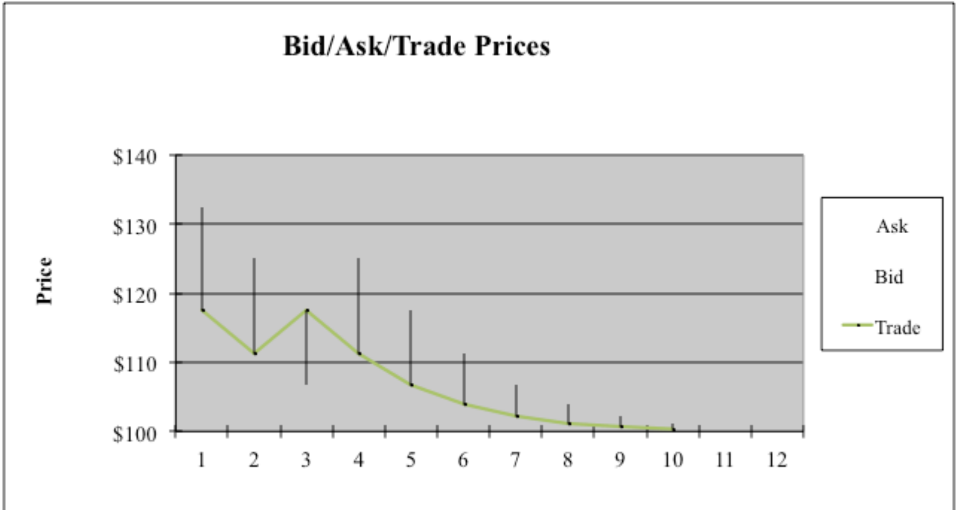
Dynamics

Ninth period: sell



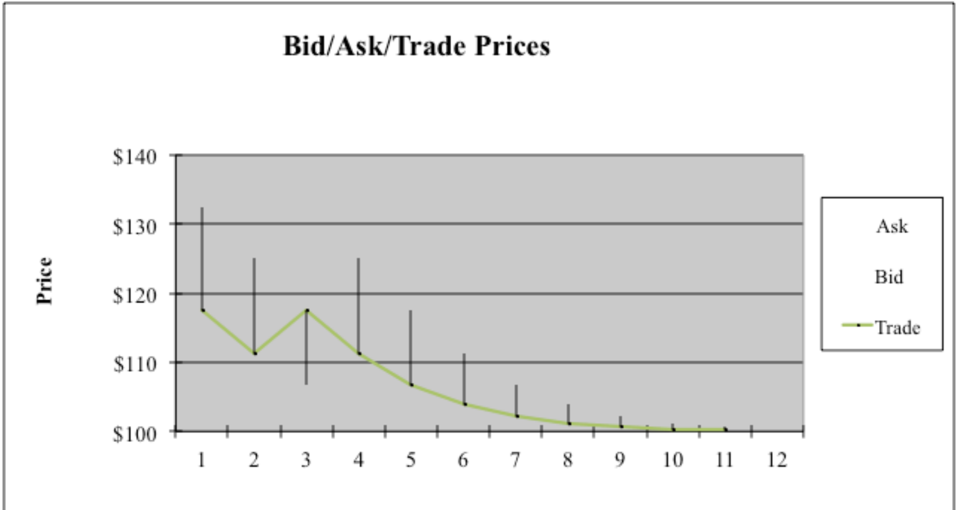
Dynamics

Tenth period: sell



Dynamics

Eleventh period: sell



Dynamics

Twelfth period: sell