

Financial Markets Microstructure

Lecture 7

Depth determinants

Chapter 4 of FPR

Egor Starkov

Københavns Universitet

Spring 2023

Previously on...

- 1 The spread is not only driven by adverse selection: order costs and inventory risk have an effect as well

Homework from last time

We said today that inventory risk is priced when the dealer is risk-averse. Risk-aversion is one explanation, but other factors can also contribute to inventory risk. The two following cases explore this issue:

- A big trader was punted off the Nordic power market after failing to meet margin calls (two articles on [absalon](#)).
 - How does inventory risk manifest in this story?
 - Explain why such inventory risk can be priced even by risk-neutral agents.
- Negative oil futures prices were registered last year (blog post on [absalon](#) or [here](#)).
 - Why did it happen? How do negative prices make sense?
 - How does inventory risk manifest in this story?
 - Explain why such inventory risk can be priced even by risk-neutral agents.

Inventory concerns more broadly

- We have discussed how dealers' risk-aversion can drive the spread, depth, and make prices deviate from the efficient level
- Two comments on that, from the cases you read:
- **Point 1:** risk-aversion in markets may stem from market risks, rather than inherent risk-aversion in preferences.
 - Standard story: $u(w)$ is concave in future wealth (e.g., MeanVar/CARA/CRRA prefs), and $w \sim z \cdot v + \dots$ (position \times asset value), hence $u(v)$ is concave in v , more so for higher z .
 - Alternative: $u(w)$ is linear (risk-neutrality), but low v creates higher risk of margin calls, which are costly: $w \sim z \cdot (v - c \cdot \mathbb{I}\{v < \underline{v}, z > 0\}) + \dots \Rightarrow u(v)$ is again concave in v .
 - Either story leads to dealer's inventory affecting their willingness to buy/sell

Inventory concerns more broadly

- **Point 2:** what if traders in the market are risk-averse, and not just the dealer?
 - If traders provide liquidity (e.g., we are in LOB market and not a dealer market) – same inventory risks
 - If general market populace is risk-averse: in a similar way, traders' valuation for the asset would depend on how far their current position is from their ideal position.
 - So if there is some aggregate imbalance – i.e., current aggregate holdings (many traders long on oil futures) are different from aggregated ideal positions (everyone wants to dump their futures) – then market price might deviate from the fundamental value
 - (Although the question to ask is: why did such discrepancy in positions arise in the first place, and is it by itself informative about the fundamentals)

■ Trade size

- How does trade size affect prices?
- I.e., what determines market **depth**?
- (Spoiler: mostly the same factors as with liquidity)
- Will look at Kyle (1985) model – an alternative to GM that allows flexible trade size

Prices and trade size

- How does trade size affect prices?
 - Spread larger for large trades, price moves further from efficient level
 - I.e., market has limited depth
- Why?

Prices and trade size

- How does trade size affect prices?
 - Spread larger for large trades, price moves further from efficient level
 - I.e., market has limited depth
- Why?
 - 1 Adverse selection: larger trades indicate more/stronger news

Prices and trade size

- How does trade size affect prices?

- Spread larger for large trades, price moves further from efficient level
- I.e., market has limited depth

- Why?

- 1 **Adverse selection**: larger trades indicate more/stronger news
- 2 **Inventory risk**: large positions are risky and take dealers longer to unwind, hence require larger premiums

Prices and trade size

- How does trade size affect prices?

- Spread larger for large trades, price moves further from efficient level
- I.e., market has limited depth

- Why?

- 1 Adverse selection: larger trades indicate more/stronger news
- 2 Inventory risk: large positions are risky and take dealers longer to unwind, hence require larger premiums
- 3 Imperfectly competitive dealers: market power allows dealers to set wider spread and steeper or flatter pricing schedules

Prices and trade size

- How does trade size affect prices?

- Spread larger for large trades, price moves further from efficient level
- I.e., market has limited depth

- Why?

- 1 Adverse selection: larger trades indicate more/stronger news
- 2 Inventory risk: large positions are risky and take dealers longer to unwind, hence require larger premiums
- 3 Imperfectly competitive dealers: market power allows dealers to set wider spread and steeper or flatter pricing schedules
- 4 Order processing costs: may increase or decrease (per stock) in total order size

Kyle model

- We will look at Kyle (1985) model which links market depth to adverse selection
- It can be extended to accomodate imperfect competition among dealers (see 4.2.4) and inventory risk (4.3)
 - the inventory risk version is broadly similar to the Stoll model that we skipped
 - trading costs are very difficult to incorporate in this model

Setup: Broad strokes

- A **call auction**; orders come from a “large” speculator and a population of noise traders; market cleared by a dealer.
- **Speculator/informed trader**: has private information
 - Trades using a ‘large’ speculative market order
 - Strategically moderates order size to reduce price impact
 - ‘Hides’ behind noise traders who submit a random size order
- Representative **market maker (MM)/dealer**
 - Risk neutral and competitive (zero profits)
 - Clears orders in *batches* (as opposed to one-by-one in Glosten & Milgrom)
 - Cannot distinguish speculative orders from noise orders in a batch

Setup

- **Asset:** Trade in one risky asset with value $v \sim \mathcal{N}(\mu, \sigma_v^2)$
- **Speculator:** Observes true value v (perfect information)
 - Places market order x
 - If the order clears at price p : gain is $x(v - p)$
 - Does **not know** p when choosing x : maximizes *expected* gain (risk neutral) given $\mathbb{E}[p|x]$
- **Noise trader:** Has random demand $u \sim \mathcal{N}(0, \sigma_u^2)$
- **MM:** Submits a supply schedule of (q, p) combinations:
 - “If the order imbalance is $q = x + u$, I will absorb it at price p ”
 - Observes aggregate flow $q = x + u$, but not x and u
 - Competitive (zero profit): $p = \mathbb{E}[v|q]$
- **Assumption:** u and v are jointly normal and independent

Setup: Timing

- To be explicit, the timing is as follows:

1 at the beginning of the period:

- speculator chooses order size x
- noise traders submit their order u
- dealer submits price schedule (q, p)

2 then market price $p(q)$ is determined given total order $q = x + u$

3 at the end of the period payoffs are realized

Linear equilibrium: outline

- The equilibrium is described by the speculator's strategy $x(v)$ and the dealer's pricing schedule $p(q)$.

Linear equilibrium: outline

- The equilibrium is described by the speculator's strategy $x(v)$ and the dealer's pricing schedule $p(q)$.
- Look for equilibrium where speculator's strategy is linear: $x = \beta(v - \mu)$
 - Note: β is endogenously determined by the equilibrium, we'll derive it
 - $\beta > 0$ measures speculator *aggression*
- MM knows the speculator's strategy (in eqm):
 - Observes $q = x + u = \beta(v - \mu) + u$, and wants to **estimate** v
 - Intuitively, $\mathbb{E}[v|q] = \mu + \lambda q$, where λ is the regression coefficient $\mathbb{C}(v, q)/\mathbb{V}(q)$, and recall that $\mu = \mathbb{E}[v]$. (This is derived formally in the following slides.)
- Since $p = \mathbb{E}[v|q]$, we can use the conditional expectation to get a price impact equation

$$p = \mu + \lambda q.$$

Here $1/\lambda$ would be a measure of market depth

Aside: Deriving the distribution of $v|q$

Claim

If $q = \beta(v - \mu) + u$ and v, u are joint normal then $v|q$ is normal, with

$$\mathbb{E}[v|q] = \mathbb{E}[v] + \frac{\mathbb{C}(v, q)}{\mathbb{V}(q)}(q - \mathbb{E}[q]),$$

$$\mathbb{V}(v|q) = 1/(1/\sigma_v^2 + \beta^2/\sigma_u^2).$$

1 Will show for our case ($v \sim \mathcal{N}(\mu, \sigma_v^2)$, $u \sim \mathcal{N}(0, \sigma_u^2)$, and $v \perp u$). In this case:

$$q \sim \mathcal{N}(0, \beta^2\sigma_v^2 + \sigma_u^2),$$

$$q|v \sim \mathcal{N}(\beta(v - \mu), \sigma_u^2).$$

Also, note that $\mathbb{C}(v, q) = \mathbb{C}(v, \beta(v - \mu) + u) = \beta\sigma_v^2$

2 Use Bayes' rule to derive the conditional density: $f(v|q) = f(v) \frac{f(q|v)}{f(q)}$.

(reminder: if $y \sim \mathcal{N}(\mu, \sigma^2)$ then its pdf is $f(y) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-\mu)^2}{2\sigma^2}}$)

$$\begin{aligned} f(v|q) &= f(v) \frac{f(q|v)}{f(q)} = \sqrt{\frac{2\pi(\beta^2\sigma_v^2 + \sigma_u^2)}{2\pi\sigma_v^2 \cdot 2\pi\sigma_u^2}} e^{-\frac{(v-\mu)^2}{2\sigma_v^2} - \frac{(q-\beta(v-\mu))^2}{2\sigma_u^2} + \frac{q^2}{2(\beta^2\sigma_v^2 + \sigma_u^2)}} \\ &= \frac{1}{\sqrt{2\pi \frac{\sigma_v^2\sigma_u^2}{\beta^2\sigma_v^2 + \sigma_u^2}}} e^{-\left[\frac{(v-\mu)^2}{2\sigma_v^2} + \frac{(q-\beta(v-\mu))^2}{2\sigma_u^2}\right] + \frac{q^2}{2(\beta^2\sigma_v^2 + \sigma_u^2)}} \end{aligned}$$

3 $f(v|q)$ looks like a normal pdf! From the leading fraction infer that $\mathbb{V}(v|q) = \frac{\sigma_v^2\sigma_u^2}{\beta^2\sigma_v^2 + \sigma_u^2}$

4 Rearrange the terms in the power to have the $2\mathbb{V}(v|q)$ in the denominator:

$$\begin{aligned}
 & -\frac{(v-\mu)^2}{2\sigma_v^2} - \frac{(q-\beta(v-\mu))^2}{2\sigma_u^2} + \frac{q^2}{2(\beta^2\sigma_v^2 + \sigma_u^2)} = \\
 & = -\frac{(v-\mu)^2\sigma_u^2}{2\sigma_v^2\sigma_u^2} - \frac{(q-\beta(v-\mu))^2\sigma_v^2}{2\sigma_v^2\sigma_u^2} + \frac{\frac{q^2}{\beta^2\sigma_v^2 + \sigma_u^2}\sigma_v^2\sigma_u^2}{2\sigma_v^2\sigma_u^2} \\
 & = \frac{1}{\frac{2\sigma_v^2\sigma_u^2}{\beta^2\sigma_v^2 + \sigma_u^2}} \left[-\frac{(v-\mu)^2\sigma_u^2}{\beta^2\sigma_v^2 + \sigma_u^2} - \frac{(q-\beta(v-\mu))^2\sigma_v^2}{\beta^2\sigma_v^2 + \sigma_u^2} + \frac{q^2\sigma_v^2\sigma_u^2}{(\beta^2\sigma_v^2 + \sigma_u^2)^2} \right]
 \end{aligned}$$

5 And then rewrite the square bracket as a parabola of v :

$$\begin{aligned}
 & = \frac{1}{2\mathbb{V}(v|q)} \left[-v^2 \frac{\sigma_u^2 + \beta^2\sigma_v^2}{\beta^2\sigma_v^2 + \sigma_u^2} + 2v \frac{\mu\sigma_u^2 + \beta^2\mu\sigma_v^2 + \beta q\sigma_v^2}{\beta^2\sigma_v^2 + \sigma_u^2} + \dots \right] \\
 & = \frac{-\left(v - \frac{\mu\sigma_u^2 + \beta^2\mu\sigma_v^2 + \beta q\sigma_v^2}{\beta^2\sigma_v^2 + \sigma_u^2}\right)^2}{2\mathbb{V}(v|q)} = \frac{-\left(v - \mu - \frac{\beta\sigma_v^2}{\beta^2\sigma_v^2 + \sigma_u^2}q\right)^2}{2\mathbb{V}(v|q)}
 \end{aligned}$$

(I leave it to you to confirm that the "... " part works out)

6 Putting everything together:

$$f(v|q) = \frac{1}{\sqrt{2\pi \frac{\sigma_v^2 \sigma_u^2}{\beta^2 \sigma_v^2 + \sigma_u^2}}} e^{-\frac{\left(v - \mu - \frac{\beta \sigma_v^2}{\beta^2 \sigma_v^2 + \sigma_u^2} q\right)^2}{2 \frac{\sigma_v^2 \sigma_u^2}{\beta^2 \sigma_v^2 + \sigma_u^2}}}$$

This is indeed a pdf of the normal distribution with

$$\begin{aligned}\mathbb{E}[v|q] &= \mu + \frac{\beta \sigma_v^2}{\beta^2 \sigma_v^2 + \sigma_u^2} q = \mathbb{E}[v] + \frac{\mathbb{C}(v, q)}{\mathbb{V}(q)} (q - \mathbb{E}[q]), \\ \mathbb{V}[v|q] &= \left(\frac{1}{\sigma_v^2} + \frac{\beta^2}{\sigma_u^2} \right)^{-1},\end{aligned}$$

as claimed.

Dealer's strategy

$$\mathbb{E}[v|q] = \mathbb{E}[v] + \frac{\mathbb{C}(v, q)}{\mathbb{V}(q)}(q - \mathbb{E}[q]) = \mu + \frac{\mathbb{C}(v, q)}{\mathbb{V}(q)}q,$$

From zero profit condition: $p = \mathbb{E}[v|q]$, hence

$$p = \mu + \lambda q,$$

where $\lambda \equiv \frac{\mathbb{C}(v, q)}{\mathbb{V}(q)} = \frac{\beta\sigma_v^2}{\beta^2\sigma_v^2 + \sigma_u^2}$

Speculator's strategy

The speculator takes for granted the pricing rule $p = \mu + \lambda q$

- The **profit** is $\Pi(x) = x(v - p) = x(v - \mu - \lambda x - \lambda u)$
- **Expected profit** is $\mathbb{E}[\Pi(x)] = x(v - \mu - \lambda x)$, since $\mathbb{E}[u] = 0$
- Speculator chooses x to maximize $\mathbb{E}[\Pi(x)]$. Using the first-order condition:

$$v - \mu - 2\lambda x = 0$$

$$\Rightarrow x = \beta(v - \mu),$$

$$\text{where } \beta = 1/(2\lambda)$$

- Confirmed that it is optimal for the speculator to use a linear strategy!
- Note analogy to monopoly problem:

Speculator's strategy

The speculator takes for granted the pricing rule $p = \mu + \lambda q$

- The **profit** is $\Pi(x) = x(v - p) = x(v - \mu - \lambda x - \lambda u)$
- **Expected profit** is $\mathbb{E}[\Pi(x)] = x(v - \mu - \lambda x)$, since $\mathbb{E}[u] = 0$
- Speculator chooses x to maximize $\mathbb{E}[\Pi(x)]$. Using the first-order condition:

$$v - \mu - 2\lambda x = 0$$

$$\Rightarrow x = \beta(v - \mu),$$

$$\text{where } \beta = 1/(2\lambda)$$

- Confirmed that it is optimal for the speculator to use a linear strategy!
- Note analogy to monopoly problem: **trade-off between trading more and trading at better price**

Closing the equilibrium

- Finally, 'match' the coefficients:

$$\frac{1}{2\beta} = \lambda = \frac{\beta\sigma_v^2}{\beta^2\sigma_v^2 + \sigma_u^2}$$

i.e. $\beta^2\sigma_v^2 + \sigma_u^2 = 2\beta^2\sigma_v^2$ which yields

$$\beta = \frac{\sigma_u}{\sigma_v} \text{ and } \lambda = \frac{\sigma_v}{2\sigma_u}.$$

- Thus: the strategies are optimal given the prices, and the prices optimal given the strategies → **equilibrium**

Equilibrium properties

$$\beta = \frac{\sigma_u}{\sigma_v} \text{ and } \lambda = \frac{\sigma_v}{2\sigma_u}.$$

- Speculator is more **aggressive** (β higher) when:

- 1 The informational advantage σ_v is smaller (why?)
- 2 There's more noise σ_u to hide behind (why?)

Equilibrium properties

$$\beta = \frac{\sigma_u}{\sigma_v} \text{ and } \lambda = \frac{\sigma_v}{2\sigma_u}.$$

- Speculator is more **aggressive** (β higher) when:

- 1 The informational advantage σ_v is smaller (why?)

- 2 There's more noise σ_u to hide behind (why?)

- **Market depth:**

$$\frac{1}{\lambda} = 2\beta = 2\frac{\sigma_u}{\sigma_v}$$

The market is deeper when there is less insider trading and more noise trading

Equilibrium properties

- Insider's a priori (before observing v) **expected gain**:

$$\begin{aligned}\mathbb{E}[x(v - \mu - \lambda x)] &= \mathbb{E}\left[\beta(v - \mu)\left(v - \mu - \frac{v - \mu}{2}\right)\right] \\ &= \beta \frac{\sigma_v^2}{2} = \frac{\sigma_v \sigma_u}{2}\end{aligned}$$

Comment: speculator expects a positive profit (could abstain). Competitive risk-neutral MM earns zero profits. Noise traders lose. Same as in GM.

Equilibrium properties

- Insider's a priori (before observing v) **expected gain**:

$$\begin{aligned}\mathbb{E}[x(v - \mu - \lambda x)] &= \mathbb{E}\left[\beta(v - \mu) \left(v - \mu - \frac{v - \mu}{2}\right)\right] \\ &= \beta \frac{\sigma_v^2}{2} = \frac{\sigma_v \sigma_u}{2}\end{aligned}$$

Comment: speculator expects a positive profit (could abstain). Competitive risk-neutral MM earns zero profits. Noise traders lose. Same as in GM.

- Market maker's perceived posterior (after observing q) **variance** of v is

$$\mathbb{V}(v|q) = \frac{1}{1/\sigma_v^2 + \beta^2/\sigma_u^2} = \frac{\sigma_v^2}{2}$$

Exactly half the prior variance: **Insider reveals half his information**

Kyle's model: summary

- **Dealer/market maker model:** Competitive, risk-neutral (zero profit) dealer chooses a supply schedule
- **Informed trader:** Observes signal about asset value and places market order
- **Market clearing:** Auction, dealer observes only total demand (informed + noise), total demand clears
- **Insights:** informed trading is a factor generating limited market depth, insider always reveals half his information
- **Advantage:** Richer trading opportunities, trader not price-taker
- **Shortcomings:** Still no resale

Kyle with inventory risk I

- Now let's look at how market maker's **inventory risk** can lead to limited depth.
- Assume **no informed trading**: $x = 0$.
- Asset value $v \sim \mathcal{N}(\mu, \sigma_v^2)$
- Market maker has mean-variance preferences over their next-period wealth:

$$U(w_{t+1}) = \mathbb{E}[w_{t+1}] - \frac{\rho}{2} \mathbb{V}(w_{t+1}),$$

where w is composed of cash and asset holdings: $w_{t+1} = (z_t - q_t)v + q_t p_t$

- MM's initial asset position is z_t (initial cash is irrelevant, ignore it).

Kyle with inventory risk II

- To get the pricing schedule, follow the competitive logic:
 - The market-maker takes some market price p as given, chooses how much $q(p)$ to sell at this price:

$$\max_q \left\{ \underbrace{(z_t - q)\mathbb{E}[v] + qp}_{\mathbb{E}[w_{t+1}]} - \underbrace{\frac{\rho}{2}(z_t - q)^2 \mathbb{V}(v)}_{\mathbb{V}(w_{t+1})} \right\}$$

- FOC: $p - \mu + \rho(z_t - q)\sigma_v^2 = 0 \iff q(p) = z_t + \frac{p - \mu}{\rho\sigma_v^2}$
- For market to clear, need $q(p) = u = q$ (dealer's supply = total traders' market order), so inverting the pricing schedule we get:

$$p(q) = \mu + \rho\sigma_v^2(q - z_t).$$

Kyle with inventory risk III

$$p(q) = \mu + \rho\sigma_v^2(q - z_t)$$

Takeaways:

- 1 Depth (dictated by the dealer's willingness to trade at a given price) is limited
- 2 This is despite traders still being completely price-insensitive in this model!
- 3 Price impact depends on asset riskiness σ_v^2 and MM's risk aversion ρ .
- 4 Midquote depends on z_t

So really, all the same stuff as in GM with inventory risk.

The book also looks at versions with many MMs with heterogeneous ρ s, and many imperfectly competitive MMs.

Other extensions are possible:

1 Dynamics

- In a fully dynamic model, the insider reveals less than half of the information in each period. Why?

Other extensions are possible:

1 Dynamics

- In a fully dynamic model, the insider reveals less than half of the information in each period. Why?
In order to better benefit from informational advantage

Other extensions are possible:

1 Dynamics

- In a fully dynamic model, the insider reveals less than half of the information in each period. Why?
In order to better benefit from informational advantage
- In the limit where trade is continuous over $[0, T]$, then $\mathbb{V}(v|q_0, \dots, q_t) \simeq (T - t) \frac{\sigma_v^2}{T}$: variance decreases linearly in time.

Other extensions are possible:

1 Dynamics

- In a fully dynamic model, the insider reveals less than half of the information in each period. Why?
In order to better benefit from informational advantage
- In the limit where trade is continuous over $[0, T]$, then $\mathbb{V}(v|q_0, \dots, q_t) \simeq (T - t) \frac{\sigma_v^2}{T}$: variance decreases linearly in time. Model of how to split a large trade over time

Other extensions are possible:

1 Dynamics

- In a fully dynamic model, the insider reveals less than half of the information in each period. Why?
[In order to better benefit from informational advantage](#)
- In the limit where trade is continuous over $[0, T]$, then $\mathbb{V}(v|q_0, \dots, q_t) \simeq (T - t) \frac{\sigma_v^2}{T}$: variance decreases linearly in time. [Model of how to split a large trade over time](#)

2 More insiders

- More insiders are more competitive; more aggressive
- The market is more liquid and more information revealed
- In dynamic model with several insiders: rush to trade on common information from the beginning

3 Imperfect market maker competition (Cournot style)

- Finite number of market makers, $k = 1, \dots, K$
- Market maker k supplies $y^k = \phi(p - \mu)$
- Market clears at price p with $\sum y^k = q$
- Strategic market maker takes into account effect of orders on profits
- Now: $p = \mu + \lambda q$ where $\lambda = \alpha(K - 1)/(K - 2) > \alpha$.

3 Imperfect market maker competition (Cournot style)

- Finite number of market makers, $k = 1, \dots, K$
- Market maker k supplies $y^k = \phi(p - \mu)$
- Market clears at price p with $\sum y^k = q$
- Strategic market maker takes into account effect of orders on profits
- Now: $p = \mu + \lambda q$ where $\lambda = \alpha(K - 1)/(K - 2) > \alpha$.

4 Trading costs

- Trivial in GM. Very difficult here, both technically and conceptually.
- Don't know how many trades there are, don't know the total volume (not q – some noise traders' orders could've cancelled each other out)
- Even taking costs as a linear function of order imbalance $|q|$ makes things difficult

Homework

- 1 We will talk about empirical estimation of factors of illiquidity next time (ch.5) and begin talking about LOB markets (without dealers; ch.6)
- 2 Solve ex 3 in ch.4 (p.159): Kyle's model with competition among speculators.