Timing of Predictions in Dynamic Cheap Talk: Experts vs. Quacks *

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Abstract

This paper studies the dynamics of announcements in prediction markets in the presence of reputation concerns. In our model, an analyst of privately known competence, who cares about his reputation, chooses when to make a prediction regarding the outcome of some future event. We find that the interplay of incentives of the quacks and the real experts produces equilibria in which the earlier reports are more credible and more informative for the receivers. Further, any report hurts the analyst's reputation in the short run, with later reports incurring larger reputation penalties. The reputation of a silent analyst, on the other hand, gradually improves over time.

Keywords: Career concerns, reputation, dynamic games, games of timing, strategic information transmission.

JEL Codes: C73, D82, D83, D84.

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1 Introduction

Where there is uncertainty, there are analysts. Build-up to any major public event summons numerous predictions of its outcome from people who claim to be experts in the field. The great recession of 2007-08 has invited plenty of forecasts as it unfolded, as well as extensive judgement of these forecasts in retrospect.[1] Every presidential election cycle in the U.S. turns much of the local and plenty of foreign media into prediction mode for many months.[2] The COVID-19 global pandemic in 2020 has seen a surge of related academic research making prognoses of possible outcomes.[3] It is no surprise, however, that not all predictions are equally informative to the receivers. The quality of a given prediction typically depends on the competence of the analyst making it – both directly, due to better insight that competence can provide, but also indirectly, due to the less competent analysts having less confidence in their reports and thus more incentives to herd with the other predictions or public information.

One way for the receivers to learn something about the quality of a prediction (and the competence of its author) is to consider *when* this prediction was made. Specifically, for a given event (macroeconomic outcomes, elections results, company earnings reports, sports outcomes, etc), earlier predictions may be more informative due to the better private information of the early analysts and/or herding effects. Alternatively, the converse may be true instead – waiting for late reports could be worthwhile if the competent analysts take their time to obtain the best possible information, while the less competent have less to learn, and so predict early [Guttman, 2010]. It is not clear at the outset which effect outweighs, and whether earlier or later reports are more informative for the receivers in the end.

Empirically, the former seems to be the case: the literature exploring earnings forecasts has repeatedly shown that the earlier analyst predictions are more informative, suggesting they are made by the more competent experts [Cooper, Day, and Lewis, 2001, Shroff, Venkataraman, and Xin, 2014, Keskek, Tse, and Tucker, 2014]. However, this effect tends to be explained via the indirect/strategic effects of competence, namely, herding behavior: once the first prediction is made, other analysts prefer to herd with this prediction and share the blame if all predictions are wrong, than to predict differently and risk large reputational damages if they end up being wrong alone [Scharfstein and Stein, 1990]. In this paper, we show that herding effects are not necessary to produce this dynamic, and earlier reports can be more informative due to direct effects alone if they are coupled with career concerns.[4] In particular, we show that if analysts differ (not in the quality of prior information, but) in the rate of accumulation of private information and choose the timing of their prediction strategically to maximize their reputation, then earlier reports are more informative than the later ones, even in the absence of any strategic considerations, such as herding effects.

Specifically, we present a model of dynamic cheap talk with career concerns. In our model,

[1] `https://www.weforum.org/agenda/2018/11/should-economists-have-foreseen-the-financial-crisis/`

[2] Many major media outlets have large and regularly updated hubs dedicated to the topic. Examples for the 2020 cycle include The Economist (`https://projects.economist.com/us-2020-forecast/president`), Financial Times (`https://ig.ft.com/us-election-2020/`), HuffPost (`https://www.huffpost.com/elections`).

[3] See Dixit [2020] for a contemporary ironic take on the issue.

[4] In doing so, we answer the call of Beyer, Cohen, Lys, and Walther [2010, p.327]: "To understand analysts' incentives, one would ideally start with their overarching objectives such as to maximize their personal utility over multiple periods (potentially including reputational or career concerns)."

an analyst, who is privately aware of his competence, makes a choice of whether and *when* to make a prediction about the outcome of some future exogenous event (state of the world). A competent analyst (an expert) gradually accumulates private knowledge about the outcome, while an incompetent analyst (a quack) has no private insight. There is no direct conflict between the analyst and the observer/receiver in our model: the analyst only cares about his reputation, while the observer only cares about the information concerning the outcome. The conflict comes from the reputational concerns within the analysts market, with the quacks trying to blend in with the experts in pursuit of reputation (and benefits that high reputation grants), preventing experts from conveying valuable information to the public.

We discover that this conflict between the quacks and the experts imposes a lot of structure on equilibrium outcomes. Our first finding is that in equilibrium, the later predictions are less informative (in terms of signal-to-noise ratio) than the earlier ones, so the quality of predictions deteriorates over the prediction cycle, consistent with the aforementioned empirical evidence. This has to be the case in order to incentivize the experts to reveal their information early, which requires that there is a higher reputational reward for early reports, which, in turn, requires that the receivers must expect early reports to be of high quality. Otherwise – if later predictions were more informative – these later predictions would be rewarded with higher reputation by the receivers. This would always make the experts prefer to delay their reports, leading to a complete prediction market shutdown.

A more surprising finding of our paper is that *all* predictions in such equilibria, although considered informative, are received with solid scepticism by the public. This is in the sense that making any prediction drops the analyst's reputation relative to what he could get by staying quiet. Moreover, the reputation penalties increase with time for as long as the analyst postpones the prediction. This phenomenon is driven by the quack's intertemporal preferences: in order to be indifferent between issuing a report today and issuing one tomorrow (where the latter, by the logic above, leads to a worse reputation), the latter option must be boosted by the high flow payoff from not making a report. Thus, silence is indeed golden in our model – silent analysts see their reputation gradually improving. Those, on the other hand, who choose to make a prediction and take a hit to their reputation, are gambling for the grand prize that is the reputation bonus for predicting the outcome correctly.

A typical path of the analyst's reputation arising from our model is illustrated in Figure 1. In this example the event occurs in period 6 and the analyst starts with reputation b_0 (which is the probability that the observer assigns to the analyst being competent). The analyst makes his report in period 4, and until then his reputation gradually increases. After the report, his reputation drops temporarily until the outcome is revealed, at which point he receives a reputation premium if his prediction turned out correct and is penalized by low reputation otherwise.

Given everything said above, it is not obvious why an analyst would ever prefer to make any prediction, i.e., take a risky gamble at the cost of short-run reputation, when staying silent would yield a risk-free high reputation. As we show, equilibria of the form described above only exist if analysts are sufficiently risk-loving or, alternatively, if gains from reputation are sufficiently convex – i.e., if the gamble of making a report is appealing enough to the quack. This is the case in, e.g., superstar markets, where the top-regarded analysts obtain a disproportionate share of consulting

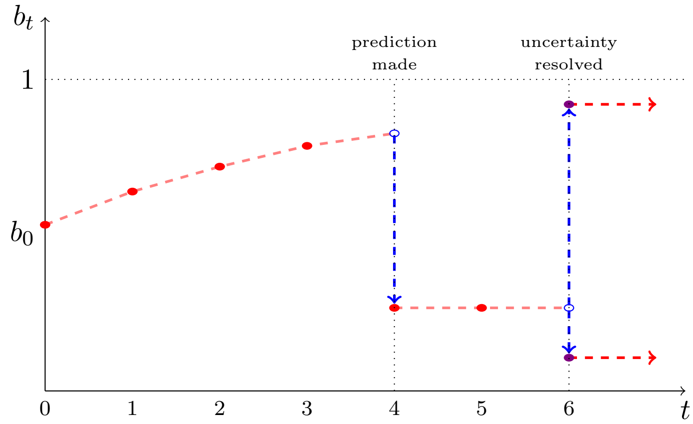

Figure 1: Analyst's example reputation path.

contracts or assets under management. Whenever this is not the case (the top reputation is not worth gambling for), only "static" equilibria exist, in which all reports are made at some single predetermined date.[5]

The paper is organized as follows. We begin by looking at a stylized example in Section 2 that presents our results in the simplest setting. Section 3 contains a review of the relevant literature. In Section 4 we formulate the general model. The main results are presented in Section 5. Section 6 contains extensions and alternative specifications. Section 7 concludes. All proofs are relegated to the Appendix.

2 Illustrative Example

This section presents an example that showcases our main results in the simplest setting. Suppose there are two periods $t = 1, 2$ and a binary state $\omega \in \{G, B\}$, which is initially not known to anybody and is publicly revealed in the end of period $T = 2$. Assume that players do not discount the future, and that the states are ex ante equally probable, i.e. $\Pr(\omega = G) = \Pr(\omega = B) = \frac{1}{2}$.

There are two players: an analyst and an observer. The analyst is, with equal probabilities, either an *expert*, or a *quack*. The analyst privately knows his type, but the observer does not. The expert has a chance λ_t to privately learn the state in period $t = 1, 2$. With positive probability, the expert also remains unaware of the state, i.e., $\lambda_1 + \lambda_2 < 1$. The quack never receives any private information about the state. In any of the two periods before the state is publicly revealed, the analyst can send one cheap talk report $m \in \{G, B\}$ to the observer, indicating his prediction about state ω. The report is not verifiable, i.e., the analyst's private information can not be made observable to the public. At the end of each period, after the report is made (or not), the analyst receives a "reputation payoff" $w(b)$. This payoff is a strictly increasing function of probability b that the observer assigns at the end of period t the analyst being an expert. For concreteness, let $w(0) = 0$.

In this example we look for an equilibrium in which the expert is honest: he reports according to his private information as soon as he obtains it and never makes an unfounded prediction or

[5] Under special assumptions there also exist degenerate equilibria, in which quack never makes any predictions for the fear of being proved wrong. See Section 6.2 for details.

reports contrary to his information.[6] How would the quack behave in such equilibrium, and how should the market react to either report and to a lack of one?

There are five actions available to the analyst in the game: he can report that the state is G or B at $t = 1, 2$, or stay silent throughout. An honest expert plays each of the five actions with a positive probability. It is immediate then that the quack must do the same in equilibrium – if either action is only taken by the expert and never by the quack, then following this path gives the analyst the highest possible reputation from that point onwards and, therefore, the highest possible continuation payoff. This would strictly dominate any alternative path of play available to the quack at the respective period.

Therefore, the quack must be indifferent between all five actions – in particular, between reporting that the state is G at $t = 1$ and $t = 2$. Denote by b_t the belief about the analyst's competence at the end of period t in case no report was made in period t; by $b(m, t)$ the belief after report m was made in period t; and by $b^{\omega}(m, t)$ the belief after report m was made in period t and the state turned out to be ω. The indifference condition between the two reports for the quack is then given by

$$w\left(b(G,1)\right) + \left[\frac{1}{2} \cdot w\left(b^G(G,1)\right) + \frac{1}{2} \cdot w\left(b^B(G,1)\right)\right] = \\ = w\left(b_1\right) + \left[\frac{1}{2} \cdot w\left(b^G(G,2)\right) + \frac{1}{2} \cdot w\left(b^B(G,2)\right)\right]. \quad (1)$$

Note that the honest expert is never wrong, since he only makes a report if he knows the state. Therefore, if the analyst made a prediction which turned out to be incorrect, he is definitely a quack: $b^B(G, 1) = b^B(G, 2) = 0$.

We have assumed that the expert always reveals his information at $t = 1$ if he has it. However, he does have an option to delay his report until the second period if he already knows the state at $t = 1$. To ensure that there is no incentive to delay, the following condition has to hold:

$$w\left(b(G,1)\right) + w\left(b^G(G,1)\right) \geqslant w\left(b_1\right) + w\left(b^G(G,2)\right). \quad (2)$$

Note that the expert's expected utility only differs from that of the quack in the probability of guessing the state correctly – the expert knows that his private signal is correct. The two expressions (1) and (2) together produce our main results described below.

Early correct reports are rewarded higher ex post. Subtracting equality (1) from (2), we immediately obtain that $b^G(G, 1) \geqslant b^G(G, 2)$. Early reports must thus be rewarded with higher reputation to incentivize the expert to reveal his information in a timely manner. Note that this only applies to reputation after the state was revealed.

Reporting harms reputation. Combining (1) with the observation above, we also infer that $b(G, 1) \leqslant b_1$, and we can similarly obtain $b(B, 1) \leqslant b_1$. Therefore, any report at $t = 1$ must decrease the analyst's reputation relative to not making a report.

[6] We show in the main analysis (Proposition 2) that this assumption is without loss of generality, since in any equilibrium the expert should be willing to report as soon as he gets the signal/information.

Reputation of a silent analyst improves. By the martingale property of beliefs, we note that $b(G,1), b(B,1)$, and b_1 must average out to b_0. The inequalities we just obtained then imply that $b_1 \geqslant b_0$: if reporting harms reputation then staying silent must improve it. With slightly more work, one can also show that $b_2 \geqslant b_1$.[7]

Early reports are more precise. The previous observations almost immediately imply that earlier reports contain more information about the state. Indeed, $b^G(G,1) \geqslant b^G(G,2)$ and $b^B(G,1) = b^B(G,2) = 0$, therefore $b(G,1) \geqslant b(G,2)$, since by the martingale property of beliefs, $b^G(m,t)$ and $b^B(m,t)$ should average out (from the observer's perspective) to $b(m,t)$. Because $b_1 \geqslant b_0$, the latter inequality means that the earlier of the two reports is relatively more likely to be made by the expert – which immediately implies that it is more informative than the latter one.

All inequalities above are generically strict. The inequalities mentioned above can only hold with equality in non-generic cases. To see this, consider a candidate equilibrium in which the quack mimics exactly the reporting strategy of the expert (bar the correlation of reports with the true state). In such an equilibrium, the expert makes no report with a positive probability since $\lambda_1 + \lambda_2 < 1$, hence so should the quack. For the quack to be indifferent at $t = 2$ between report $(G,2)$ and no report (with the latter yielding stable reputation regardless of state realization), it should hold that

$$w(b_2) = \frac{1}{2} \cdot w\left(b^G(G,2)\right) + \frac{1}{2} \cdot w\left(b^B(G,2)\right)$$
$$\Rightarrow b_2 = \frac{1}{2} \cdot b^G(G,2),$$

where $b^G(G,2) = \frac{b_1}{b_1 + \frac{1}{2}\cdot(1-b_1)}$ and by assumption, $b_2 = b_1$. The system above has no solution w.r.t. b_2 (unless $b_2 = b_1 = b_0 = 0$, which is not an interesting scenario), hence such a candidate equilibrium cannot exist. It then follows immediately that in any equilibrium in this example (if any such equilibria exist), early correct reports are rewarded *strictly* higher ex post, reporting *strictly* harms reputation, reputation of a silent analyst *strictly* improves, and early reports are *strictly* more precise.

Quacks only enter superstar markets. The construction above implies that the quack faces a choice between improving his reputation by staying silent ($b_2 \geqslant b_1 \geqslant b_0$) and making a report in either period, which entails a reputational gamble with a lower mean ($b(G,1) \leqslant b_0$). In order for the latter to be a viable strategy, high payoff $w(b^G(G,1))$ from a high reputation after a correct prediction must significantly outweigh a possible loss after an incorrect prediction. In other words, there must exist a disproportionate premium for becoming a highly-regarded analyst in order for the quacks to participate in the market.[8]

The remainder of the paper expands on the analysis of this example in a general framework and shows that the insights demonstrated above are quite general.

[7]By the martingale property, $b(G,2), b(B,2)$, and b_2 must average out to b_1, and we know that $b(m,2) \leqslant b(m,1) \leqslant b_1$ for $m = G, B$. Therefore, $b_2 \geqslant b_1$.

[8]If this does not hold, then only "static" equilibria exist, in which all analysts coordinate on issuing their reports in the same period.

3 Relation to the Literature

This paper mainly contributes to two strands of literature: communication with *career concerns* and the *timing of communication*.

The importance of career concerns for informative communication was first argued by Holmström [1999]. One of Holmström's original examples illustrates that an analyst may be reluctant to truthfully reveal his private information for fear of making a mistake and appearing incompetent, preferring instead to herd with public information or reports of other analysts.[9] Other papers have argued that some cohort of analysts – or even all of them in some settings – may, conversely, resort to extreme reports, overstating their private signals in order to separate themselves from "herders" (see Prendergast and Stole [1996], Graham [1999], Hong, Kubik, and Solomon [2000], Lamont [2002], Ottaviani and Sørensen [2006b], Mariano [2012], Kang and Kim [2022]). Either way, it is generally agreed that analysts' career concerns make information transmission noisy.[10] Dewatripont, Jewitt, and Tirole [1999], Prat [2005], Ottaviani and Sørensen [2006c], Rodina [2020], and Vong [2023] give various general characterizations of communication outcomes in the presence of career concerns and their dependence on the information structure of the game.

Of all papers mentioned above only a few look at the dynamics of announcements. In the model of Prendergast and Stole [1996] the analyst obtains his private information gradually over time, and his competence determines the speed of learning. They establish that the analysts overreact to early pieces of information in order to establish their reputation for competency early on, while as time progresses, they become too reluctant to change their decisions and thus underreact to late information. Predictions of a model by Graham [1999] can be interpreted in a similar way.[11] Hong, Kubik, and Solomon [2000] and Lamont [2002] find a completely opposite pattern in the data: as the analysts become older and more established, they usually make more extreme predictions. Li [2007] shows theoretically that when an analyst acquires multiple pieces of information over time, changing one's prediction can act as a signal of competence. However, timing of the prediction or a decision is never a choice variable for the analyst in these papers. Our paper fills this gap by examining how an analyst can manipulate his reputation by strategically choosing the timing of his prediction. It is also worth noting that much of the career concerns literature assumes analysts are uninformed of their type, whereas we focus instead on the case where the analyst is privately informed about his competence.

Keskek, Tse, and Tucker [2014] provide evidence from the field that competent experts tend to make their reports earlier – so earlier reports are more informative and are perceived more favorably, – and explain this through preemption mechanisms. We show that competition is *not* necessary for this phenomenon to arise. Bernhardt, Campello, and Kutsoati [2006] also explore competitive prediction markets and discover strong anti-herding dynamics in the data.

[9] This idea was picked up and greatly extended upon by the literature that followed: see Scharfstein and Stein [1990], Trueman [1994], Ely and Välimäki [2003], Ottaviani and Sørensen [2006a], Dasgupta and Prat [2008], Andina-Díaz and García-Martínez [2020].

[10] Zábojnik [2001], Ely and Välimäki [2003] and Klein and Mylovanov [2017] argue that if all analysts are patient enough, then this noise vanishes, and communication efficiency is restored. Backus and Little [2020] and Tajika [2021] show that making analysts admit uncertainty (not knowing the answer) is not a trivial problem either in the presence of career concerns.

[11] Bernhardt, Wan, and Xiao [2016] observe inertia in financial analysts' predictions, but their explanation of this phenomenon does not rely on career concerns.

The second large (and growing) strand of literature this paper contributes to is that on dynamic communication and, especially, the timing of communication.[12] Guttman, Kremer, and Skrzypacz [2014] provide a notable illustration on the importance of timing in communication in the context of dynamic disclosure. They show that the same piece of hard (verifiable) information can induce different reactions when disclosed at different times. Possibly the most relevant to ours is the contemporary paper by Boleslavsky and Taylor [2020]. They consider a timing game, in which an agent decides when to propose his project to the principal. The agent wants the project to be accepted, while the principal wants the project to be finished/ready by the time it is proposed. They assume an honest agent proposes as soon as the project is ready (which happens at a random time), and an opportunistic agent chooses whether to do the same or to offer a fake/unfinished project – and if so, when to do it. Their model yields some predictions that are similar to ours, e.g., that reputation of a non-proposing agent increases over time, but differs substantially in payoff structure and in terms of focus (they mainly explore measures that can help the principal screen the projects).

In other related papers, Guttman [2010], Acharya, DeMarzo, and Kremer [2011], Gratton, Holden, and Kolotilin [2017], and Aghamolla and An [2021] also investigate optimal timing in the context of dynamic disclosure of verifiable information. In contrast to these papers, we deal with soft information, which cannot be credibly disclosed. Grenadier, Malenko, and Malenko [2016] study a setting in which the informed expert uses the timing of his (non-verifiable) report to manipulate the timing of the observer's decision. A separate literature explores dynamic revelation of static information and finds, to some surprise, that even if all agents possess all of their respective information in period zero, allowing for multi-period communication may sometimes allow for higher payoffs to some or all parties.[13] All of the aforementioned communication models assume direct conflict of interest between the sender(s) and the receiver(s). Our model of career concerns is different in this regard, since all barriers to truthful communication stem instead from the conflict within the senders' market, namely between competent and incompetent analysts.[14]

Finally, our paper takes the market for predictions as given rather than designing it in such a way as to extract the most information from the analyst. A general approach to dynamic mechanism design when experts have evolving private information has been proposed by Pavan, Segal, and Toikka [2014]. A sub-field of mechanism design explicitly deals with optimal statistical testing of experts' competence (knowledge of a signal-generating process): see Olszewski [2015] for a recent survey.[15] Chambers and Lambert [2021] design mechanisms for dynamic belief elicitation. Our paper is different from this literature in that it does not give the observer the power to design payoffs or information feedback. Instead, it asks the question of whether market forces alone can enable informative communication.

[12] For dynamic models of *repeated* communication, where the sender does not have the choice of timing, see Sobel [1985], Bénabou and Laroque [1992], Morris [2001], Alizamir, de Véricourt, and Wang [2020], Pavesi and Scotti [2022].

[13] See Aumann and Hart [2003], Krishna and Morgan [2004], Chen, Goltsman, Hörner, and Pavlov [2017], Lipnowski and Ravid [2020], and Alonso and Rantakari [2022].

[14] Curiously, effects similar to career concerns models can be obtained in communication settings with sender-receiver conflict where the sender's deceit can be detected with positive probability. For examples of such models see Dziuda and Salas [2019] and Drugov and Troya-Martinez [2019].

[15] The most recent contributions to this field include Pei [2016], Deb, Pai, and Said [2018], Ginzburg [2020], and Smolin [2021].

4 The Model

4.1 Primitives

Time is discrete and finite: $t \in \{0\} \cup \mathcal{T}$ where $\mathcal{T} \equiv \{1, \dots, T\}$ for some $T > 0$. An underlying standard probability space is implied throughout the paper. The probability measure on this space is denoted by P.

State of the world. There is a binary state of the world ω which can be either *good* or *bad*: $\omega \in \{G, B\}$. The commonly held prior belief that the state is good is $P(\omega = G) = p_0 \in \left[\frac{1}{2}, 1\right)$. Initially the state is uncertain; at the end of period T the state is revealed.

Players. There are two players: an observer (she) and an analyst (he). Both players live for T periods and do not discount the future.

The analyst has a binary type $\gamma \in \{E, Q\}$: he can be competent or incompetent or, as we call them, an expert (E) or a quack (Q) respectively. The type is privately known by the analyst, but is not known by the observer. The observer's prior belief that the analyst is competent is $b_0 \in (0, 1)$.

The observer has no actions in the model.[16] In every period t she updates her beliefs p_t about the state of the world and b_t about the analyst's competence. It will prove convenient to represent these beliefs as likelihoods rather than probabilities, so let $\rho_t := \frac{p_t}{1-p_t}$ and $\beta_t := \frac{b_t}{1-b_t}$.

The analyst starts with no private information, regardless of type: $\eta_0^\gamma = \varnothing$ for both γ. At some random time $t^* \sim F(t)$, the competent analyst observes a private signal about the state: $\eta_{t^*}^E \in \{G, B\}$. We let $\eta_t^\gamma \in \{\varnothing, G, B\}$ denote the analyst's private information about the state at time t. The expert's private signal has precision $\pi := P(\eta_{t^*}^E = G | \omega = G) = P(\eta_{t^*}^E = B | \omega = B)$. For most of the paper we assume $\pi = 1$, but in Section 6.3 we show that all results continue to hold in case of imperfect signals, $\pi \in \left(\frac{1}{2}, 1\right)$, given some extra conditions. We assume that $F(t)$ is a c.d.f. with full support on $\mathcal{T}$ and that $F(T) < 1$, i.e., there is a positive probability that the signal arrives at any time t, and it is also possible that it never arrives.[17] We denote the conditional probability (hazard rate) of signal arrival in period t as $\lambda(t) := \frac{F(t)-F(t-1)}{1-F(t-1)}$.

The analyst receives a per-period "reputation payoff" $w(\beta_t)$ which depends on the observer's belief about the analyst's competence held at the end of period t. We assume $w(\cdot)$ is strictly increasing. As a normalization, let $w(0) = 0$. After the state is revealed, the analyst receives a terminal payoff $w^c(\beta_T)$, representing the analyst's continuation value from the reputation he has accumulated. We assume that $w^c(\cdot)$ satisfies the same requirements that we impose on $w(\cdot)$. Payoffs are interpreted as coming from some external source rather than the observer directly: a highly regarded analyst can bargain higher wage from employers in the labor market or get more followers on social media.

Communication. In any period $t \in \mathcal{T}$, the analyst can send a report $m \in \{G, B\}$ to the observer, indicating his prediction about state ω, or stay silent. The report is not verifiable, i.e., the analyst's private information is not ever observable and/or contractible. Additionally, we assume that the analyst can send at most one report throughout the game.[18]

[16] In the discussion surrounding the model, we assume that she is interested in information about state. To fix ideas, one may think that the observer chooses a binary action from $\{G, B\}$ at time T and receives a fixed reward if and only if her action matches the state – but we do not model this decision explicitly.

[17] To make $F(t)$ a proper c.d.f. in case $F(T) < 1$, one can think that with probability $1 - F(T)$, the signal arrives in a ficticious period $T + 1$, at which point it cannot be utilized by the analyst, and so $F(T + 1) = 1$.

[18] This constraint should not be seen as restrictive since the analyst receives at most one private signal by time T

4.2 Timing

In period $t = 0$, the state of the world ω and the analyst's type γ are realized, and the expert's private signal realization $\eta^E_{t^*}$ and signal arrival time t^* are drawn from the respective distributions. After that, in every period $t \in \{1, ..., T-1\}$, the stage game proceeds as follows:

1. Private information η^γ_t is updated (i.e., if $\gamma = E$ and $t = t^*$, the analyst gets a private signal);
2. The analyst updates his belief about the state conditional on η^γ_t and decides whether to send a report m to the observer;
3. The observer updates her beliefs, ρ_t about the state and β_t about the analyst's competence, conditional on the analyst's report or lack of thereof;
4. The analyst receives payoff $w(\beta_t)$;

In period T steps 1 and 2 take place as above, but instead of steps 3 and 4 the following happens:

3. State ω is publicly revealed;
4. All players update their beliefs accordingly;
5. The analyst receives a terminal lump-sum payoff $w^c(\beta_T)$.

4.3 Histories and State Variables

A *message history* is $\mu_t = (m, s)$ if report m has been made in period $s \leqslant t$ and $\mu_t = \varnothing$ otherwise. A *public history* h^p_t is a tuple consisting of the variables that are publicly observable at the beginning of period t: $h^p_t = (t, \mu_{t-1})$. The analyst possesses private information about his type and his private signal in addition to whatever is publicly known. We define a type-γ analyst's *private history* as $h^\gamma_t = (t, \mu_{t-1}, \eta^\gamma_t)$.[19] The quack never receives any private signals, so $\eta^Q_t = \varnothing$ for all t. The quack's private histories are then equivalent to public histories, and hereinafter we will treat them as such. We also let $-\eta$ and $-m$ denote the "opposites" of η and m respectively: e.g., if $\eta = G$ then $-\eta = B$.

4.4 The Analyst's Problem

At every history, the analyst decides whether to send a report and, if yes, which report to send. The analyst's pure strategy is thus a mapping from private histories h^γ_t to the set of feasible messages (which equals $\{\varnothing, G, B\}$ if no report has yet been made and $\{\varnothing\}$ otherwise, since we restrict analyst to sending at most one report). The analyst's mixed strategy is, as usual, a probability distribution over pure strategies.

Since only histories with $\mu_{t-1} = \varnothing$ involve a non-trivial choice of message, we can define strategies on the smaller space of tuples (t, η) instead of the space of all private histories $h^\gamma_t =$

and there are no public signals.

[19] In principle, the informed expert's private history also contains t^*, the arrival time of the private signal. By omitting t^* from h^E_t, we restrict the informed expert's strategies to be independent of t^*. It is, however, straightforward that this restriction is without loss of generality, because t^* is not payoff-relevant to any player, and neither is it observable by anyone except the expert, so it can be seen as nothing more than the expert's private randomization device.

$(t, \mu_{t-1}, \eta_t^\gamma)$. Therefore, we introduce the analyst's *behavioral strategy* as $r_\eta^\gamma(m,t)$, which denotes the probability of analyst γ making report m at time t conditional on having private information $\eta_t^\gamma = \eta$ and having not made any report prior to t. Finally, let $r^\gamma(m,t) := \mathbb{E}_\eta r_\eta^\gamma(m,t)$ denote the hazard rate of report (m,t) by an analyst of type γ as perceived by the observer who does not possess the analyst's private information η.

The analyst's optimization problem is hence as follows: at every private history h_t^γ such that no report has yet been made ($\mu_{t-1} = \varnothing$), the analyst of type $\gamma \in \{E, Q\}$ chooses a contemporaneous reporting strategy $r_\eta^\gamma(m,t)$ (and a sequentially rational plan for future reports) as a solution to the following problem:

$$V_{t,\eta}^\gamma := \max_{r_\eta^\gamma} \mathbb{E}\left[\sum_{s=t}^{T-1} w\left(\beta(h_s^p)\right) + w^c\left(\beta(h_T^p)\right) \middle| t, \eta, \mu_{t-1} = \varnothing\right] \tag{3}$$

subject to the evolution of $\beta(h_s^p)$ as described in the following subsection. The expectation is taken over all possible future histories.

As the quack never receives the private signal, we supress the subscript η when talking about $V_{t,\eta}^Q$ and refer to it as simply V_t^Q.

4.5 Beliefs

The two important characteristics of any public history $h_t^p = (t, \mu_{t-1})$ are the observer's beliefs about the type of the analyst and about the state of the world at that history, $\beta(h_t^p)$ and $\rho(h_t^p)$. It will prove convenient to introduce the following shorthand labels for beliefs:

$$\beta(m,t) := \beta(\tau, (m,t)) \qquad \rho(m,t) := \rho(\tau, (m,t))$$
$$\beta_t := \beta(t, \varnothing) \qquad \rho_t := \rho(t, \varnothing)$$

for all $\tau \geqslant t$. In this notation, $\beta(m,t)$ is the belief about the analyst's type held by the observer at any time $\tau \geqslant t$ conditional on report m made at time t, and β_t is the belief held at time t in the absence of any reports. Note that $\beta(m,t)$ is well defined, because once a report has been made, the observer's belief is frozen in place since no further information can be conveyed from the analyst to the public. The same applies to the observer's belief about state, ρ, and we will use the same notation for the respective probabilities b and p whenever applicable.

Finally, we let $\beta^\omega(m,t)$ denote the belief about the analyst's type given a terminal history $h_T^p = (T, (m,t))$ and given that the state was revealed to be ω.

4.6 Equilibrium Definition

We are looking for Weak Perfect Bayesian Equilibria of the game, which consist of a strategy profile $\{r_\eta^\gamma(m,t)\}$ and a belief profile $(\beta(h_t^p), \rho(h_t^p))$ such that:

1. strategies r_η^γ solve (3) given the observer's updating rule for $\beta(h_t^p)$,

2. all players update their beliefs via Bayes' rule on path,

We further adopt three following refinements, which are implied whenever we talk about an "equilibrium", unless stated otherwise:

(OP) Off-path Pessimism: off the equilibrium path the beliefs are $\rho = \rho_0$ and $\beta = 0$, with the exception that an extreme belief $\beta = +\infty$ ($b = 1$) is not updated,

(ML) Message Labeling: either there exists $\eta \in \{G, B\}$ such that $r^E_\eta(\eta, t) > r^E_\eta(-\eta, t)$, or $r^E_G(G, t) = r^E_G(B, t)$ and $r^E_B(G, t) = r^E_B(B, t)$,

(SY) Symmetry: $r^E_G(G, t) = r^E_B(B, t)$.

Off-path pessimism (OP) only makes it easier to sustain any given strategy profile as equilibrium because it makes deviations extremely unappealing for the analyst. In particular, if there is some PBE with some off-equilibrium path beliefs, then the same profile of strategies and on-path beliefs would still constitute a PBE when paired with the off-path beliefs prescribed by (OP). The exception in (OP) comes into play only at specific histories if $\pi = 1$.[20]

Message labeling (ML) requires that report m is more indicative of state $\omega = m$ than the other report. This assumption is without loss, since at any history h^p_t we can assign message labels G and B to the two messages in such a way that (ML) is satisfied.

The only requirement that imposes any actual restrictions is symmetry (SY). It requires that the expert treats states and messages equally – if he has evidence of state G, he sends report G with the same probability that he would have sent report B if he had evidence of state B. This assumption is made primarily for tractability: in case of asymmetric equilibria, even formulating the results becomes a daunting task.

5 Equilibrium Analysis

Cheap talk games like ours are infamous for equilibrium multiplicity problems, and characterizing all Weak PBE is not a trivial task. While some of them are refined away by the restrictions imposed above, a lot of equilibria remain. One source of multiplicity is the self-reenforcing restriction on the set of reports sent in equilibrium. In particular, it is easy to see that if some report $m \in \{G, B\}$ is off the equilibrium path in period t, then restriction (OP) guarantees that indeed, it is not optimal for any analyst type to send it at t. The main question that is answered in this section is as follows: assuming that in equilibrium, reports are only made at some set of periods $S \subseteq \mathcal{T}$, how do the analyst's strategies look and how does the informativeness of the reports change across different periods? It turns out that all equilibria have quite a lot of common structure. Section 5.5 then provides a partial characterization of when the respective equilibria exist.

Proofs of all statements presented in this chapter can be found in the Appendix.

5.1 Belief Updating

This section specifies how exactly the observer's belief β about the analyst's type evolves given the analysts' strategy profile $\{r^\gamma_\eta(m, t)\}$. The observer's belief ρ about the state of the world is not directly relevant to the analysis, hence the expressions related to it are relegated to the Appendix.

[20] The exception is known as a "never dissuaded once convinced" assumption in the signaling and bargaining literature (see Starkov [2023] for a broad discussion of this assumption in dynamic games). As the name suggests, it requires that once the observer makes up her mind (converges to a degenerate belief), she does not update her belief further. This can be interpreted as a limit of the model as $\pi \to 1$. Section 6.2 discusses alternatives to (OP) in case $\pi = 1$.

Conditional on the analyst not making a report, the observer's beliefs are updated as follows:

$$\beta_t = \beta_{t-1} \cdot \frac{1 - r^E(G,t) - r^E(B,t)}{1 - r^Q(G,t) - r^Q(B,t)}. \tag{4}$$

Conditional on no report by the end of period T and realized state ω, the observer's terminal belief is

$$\beta^\omega(\varnothing) = \beta_0 \cdot \prod_{t=1}^{T} \left(\frac{1 - \mathbb{E}_\eta \left[r_\eta^E(G,t) + r_\eta^E(B,t) | \omega \right]}{1 - \left[r^Q(G,t) + r^Q(B,t) \right]} \right). \tag{5}$$

Similarly, employing the Bayes' rule we can derive the observer's belief $\beta(m,t)$ following the analyst's report (m,t), and the observer's terminal belief $\beta^\omega(m,t)$ given report (m,t) and the realized state ω:

$$\begin{aligned} \beta(m,t) &= \beta_{t-1} \cdot \frac{r^E(m,t)}{r^Q(m,t)}, \\ \beta^\omega(m,t) &= \beta_{t-1} \cdot \frac{\mathbb{E}_\eta \left[r_\eta^E(m,t) | \omega \right]}{r^Q(m,t)}. \end{aligned} \tag{6}$$

5.2 Supports of the Reporting Times

Given $\gamma \in \{E, Q\}$, define support $\mathcal{S}^m$ as the set of times t at which report m can be made:[21]

$$\mathcal{S}^m := \{ t \in \mathcal{T} \mid r^\gamma(m,t) > 0 \text{ for some } \gamma \}. \tag{7}$$

Further, let $\mathcal{S} := \mathcal{S}^G \cup \mathcal{S}^B$. Denote the periods in the support as $\mathcal{S} = \{t_1, t_2, \ldots, t_{|\mathcal{S}|}\}$. The following proposition states that the message supports are common for the two analyst types.

Proposition 1. *In any equilibrium, any report (m,t) for $m \in \{G, B\}$ is made with positive probability by a quack if and only if it is ever made by an expert: $r^E(m,t) > 0 \iff r^Q(m,t) > 0$.*

The reasoning behind this proposition is as follows. Suppose that there exists a report (m,t) such that $r^Q(m,t) > 0$ but $r^E(m,t) = 0$, i.e., report m at t is only ever made by a quack. Then after report (m,t), the observer infers that the analyst is surely incompetent. This renders report (m,t) to be a dominated reporting strategy for the analyst – strictly so if we recall that the belief about the analyst's type is a martingale. Therefore, there must exist another continuation strategy at time t – i.e., at the public history $h_t^p = (t, \varnothing)$ – that results in strictly positive reputation for at least one period. No analyst is willing to play a strictly dominated strategy, hence this cannot happen in equilibrium. A similar logic is in play in the opposite case – if $r^E(m,t) > 0$ and $r^Q(m,t) = 0$ for some (m,t) – except then reporting (m,t) is a strictly dominant strategy for any type of the analyst since it yields the maximal possible reputation starting from t for the rest of the game. Reporting (m,t) is then strictly preferred by the quack to any other alternative, which again yields a contradiction.

Proposition 1 together with (OP) clearly imply that the quack should be indifferent between all reports if they are made with a positive probability from the observer's viewpoint.

[21] More formally, message support $\mathcal{S}^m$ is a subset of public histories h_t^p for which $r^\gamma(m,t) > 0$ for some γ. Since a public history in our model consists of current time t and a messaging history μ_t, and reports can only be made at histories with $\mu_t = \varnothing$, it is without loss to define the support simply as a set of times.

5.3 Informative Reports and Babbling

If a report (m, t) is made in equilibrium, this does not by itself mean that it contains any meaningful information about the state of the world or the type of the analyst. Following Crawford and Sobel [1982], we refer to such uninformative reports as *babbling*.

Definition. *A report* (m, t) *is* babbling *if both of the following hold:*

$$\beta(m, t) = \beta_{t-1}, \tag{8}$$

$$\rho(m, t) = \rho_{t-1}. \tag{9}$$

A report (m, t) *is* informative *if it is not babbling.*

Condition (8) implies that the report is uninformative about the analyst's type, while (9) implies that it contains no information about the state.

Due to the restriction that the analyst can send at most one report, babbling reports in any equilibrium are organized in a specific structure. This is illustrated by the following proposition.

Proposition 2. *Every equilibrium contains a* Godwin point $\bar{t} := \min\{t \in \mathcal{T} \mid V^E_{t,\varnothing} = V^Q_t\}$ *such that:*

1. *All on-path reports* (m, t) *with* $t > \bar{t}$ *are babbling.*
2. *No on-path reports* (m, t) *with* $t \leqslant \bar{t}$ *are babbling. Moreover:*
   - *at every* $t < \bar{t}$ *the expert does not make a report unless he has received the corresponding signal, i.e.,* $r^E_\varnothing(m, t) = 0$ *and* $r^E_\eta(m, t) = 0$ *whenever* $\eta \neq m$*;*
   - *at* $t = \bar{t}$ *the informed expert always reports his signal, i.e.,* $r^E_\eta(\eta, \bar{t}) = 1$*.*

In popular culture, the "Godwin's law" states that as a discussion on the Internet continues for long enough, the probability of a comparison involving Nazis or Hitler approaches 1.[22] At that point the informative part of the discussion is usually considered finished, and what follows is just babbling. Along similar lines, Proposition 2 says that in our model all equilibria feature at most two phases: in the early phase, the reports are informative, while in the later phase the reports do not contain any relevant information about the state or the type of the analyst.

To understand Proposition 2 it is enough to note that by Proposition 1, $t \in \mathcal{S}$ only if an expert is willing to report at t. His comparative advantage relative to quack is his ability to acquire private signals. Therefore, the expert is only willing to participate in babbling if he has no option to exploit his [current or possibly future] information by sending an informative report – i.e., if the Godwin point $\bar{t}$ has passed and the discourse has descended into babbling. Conversely, whenever an option to make an informative report now or in the future is present ($t < \bar{t}$), the expert would not be willing to use his only chance to make an impact to make an unfounded report (or report contrary to his private information).

It is worth noting that $\bar{t}$ does not have to be in the interior of the support, so one of the phases may be absent: if $\bar{t} < t_1$ then all reports are babbling, while if $\bar{t} = t_{|\mathcal{S}|}$ then no babbling takes place in equilibrium. We shall refer to the latter type of equilibria as *informative*.

[22]See `https://www.wired.com/1994/10/godwin-if-2/`.

Definition. *An* informative equilibrium *is an equilibrium where all reports in the support are informative.*

Note that in any informative equilibrium with support $\mathcal{S}$, it must be that $\bar{t} = t_{|\mathcal{S}|}$, since the definition directly implies $\bar{t} \geqslant t_{|\mathcal{S}|}$, and the condition $V^E_{t,\varnothing} = V^Q_t$ is satisfied for $t = t_{|\mathcal{S}|}$. The next proposition shows that the babbling phase may be safely ignored altogether, and without loss of generality we can restrict attention to informative equilibria.

Proposition 3 (Babbling Irrelevance)**.** *For any equilibrium with support $\mathcal{S}$ and Godwin point $\bar{t}$, there exists an informative equilibrium with the same Godwin point $\bar{t}$ and support $\tilde{\mathcal{S}} = \mathcal{S} \cap \{t \leqslant \bar{t}\}$ such that the two equilibria are:*

1. *payoff-equivalent for all players,*

2. *strategy-equivalent on $\tilde{\mathcal{S}}$.*

Propositions 2 and 3 together imply that any equilibrium strategy profile with some Godwin point $\bar{t}$ can be obtained from a respective informative equilibrium with the same Godwin point by allowing for some babbling in periods $\{\bar{t}+1, ..., T\}$.

Finally, to simplify the statements of our results, we will also focus on *reticent* equilibria, in which the expert never makes any unfounded reports.

Definition. *An equilibrium is* reticent *if $r^E_\varnothing(G, \bar{t}) = r^E_\varnothing(B, \bar{t}) = 0$.*

The expert in such equilibria only makes a prediction m if he has received private signal $\eta = m$. Proposition 2 established that this must be true for all $t < \bar{t}$, so the restriction only applies to the Godwin point $t = \bar{t}$. The restriction is needed exactly so that all points in the support can be treated equally, without the Godwin point requiring individual statements. Section 6.1 shows how our results extend to equilibria that are not reticent.

We now proceed to our main characterization result.

5.4 Main Results

This section fixes an arbitrary support $\mathcal{S} = \{t_1, t_2, \ldots, t_{|\mathcal{S}|-1}, t_{|\mathcal{S}|} = \bar{t}\} \subseteq \mathcal{T}$ and explores the properties of the informative reticent equilibria on $\mathcal{S}$ (assuming they exist). Other kinds of equilibria, including the non-reticent ones, are explored in Section 6. For simplicity we also assume throughout the remainder of Section 5 that the expert's signals are absolutely precise ($\pi = 1$); this assumption is relaxed in Section 6.3.

To talk about the informativeness of different predictions about the state of the world, we introduce the following measure:

$$i(m, t) := \ln(\rho(m, t)) - \ln(\rho_{t-1}). \tag{10}$$

This measure shows how likely report (m, t) is to be sent in state G as opposed to state B. Positive values reinforce the observer's belief in state $\omega = G$ after hearing this report, while negative values do the same for state $\omega = B$. Higher absolute values of $i(m, t)$ mean that more information is transmitted by message (m, t) to the observer, meaning that belief $\rho(m, t)$ moves further away from

ρ_{t-1}. Note that since the Bayes' rule is linear in log-likelihoods, $|i(m,t)|$ shows exactly the "strength" of the signal contained in (m,t) in terms of its effect on the posterior belief $\rho(m,t)$ relative to the prior ρ_{t-1}. This measure is selected due to it being simple and intuitive, but the specific functional form is not important for the results. Namely, we show in Section 6.4 that all of the results continue to hold if we replace $i(m,t)$ with the Kullback-Leibler divergence of $\rho(m,t)$ from ρ_{t-1}.

Presented next is the central result of our paper, which describes the informational content of reports and the informativeness dynamics. All monotonicity statements in this Theorem are understood in the sense of weak monotonicity.

Theorem 1. *Fix some $\mathcal{S}$ with $|\mathcal{S}| \geqslant 2$. Suppose that $\pi = 1$ and an informative reticent equilibrium on $\mathcal{S}$ exists. Then in any such equilibrium the following are true for both $m \in \{G, B\}$:*

1. *later reports are less informative about the state: $|i(m,t)|$ is a decreasing function of t on $\mathcal{S}^m$;*
2. *the reputation of a silent analyst improves over time: β_t is increasing in t on $\mathcal{S}$ and constant on $\mathcal{T}\backslash\mathcal{S}$;*
3. *making any report decreases reputation as compared to no report: $\beta(m,t) \leqslant \beta_t$ for any $t \in \mathcal{S}$.*

Theorem 1 states that in any reticent informative equilibrium with $|\mathcal{S}| \geqslant 2$, the reports should become [weakly] noisier over time. This is required to provide incentives for the expert to disclose the information he possesses. To elaborate, Proposition 1 implies that a quack must be indifferent between all reports (m,t) made in equilibrium. The only difference between the expert's and the quack's payoffs comes from their respective probabilities of guessing the state correctly with their report. Therefore, conditional on the quack being indifferent between all reports, the expert with information $\eta \in \{G, B\}$ in period t effectively maximizes the net premium for guessing the state correctly, as given by

$$\varDelta w_\eta(m,\tau) := w^c\left(\beta^\eta(m,\tau)\right) - w^c\left(\beta^{-\eta}(m,\tau)\right), \tag{11}$$

over all reports (m,τ) with $\tau \geqslant t$. From Proposition 2 we know that it is enough to consider reports that coincide with the expert's information: $m = \eta$. Moreover, Propositions 1 and 2 together imply that in informative equilibria, $t \in \mathcal{S}$ if and only if an informed expert reports at t. This means that for $t \in \mathcal{S}$ we have

$$(\eta, t) = \arg \max_{(m,\tau)\in\mathcal{S}|\tau\geqslant t} \varDelta w_\eta(m,\tau)$$

or, simply speaking, $\varDelta w_\eta(m,t)$ must be a weakly decreasing function of t on $\mathcal{S}$ for $m = \eta$ to incentivize the expert to report sooner, rather than later. Note that in case $\pi = 1$, Proposition 2 implies that $\beta^{-\eta}(\eta,t) = 0$ for all $t \in \mathcal{S}\backslash\{\bar{t}\}$, and therefore, $\varDelta w^c_\eta(\eta,t) = w\left(\beta^\eta(\eta,t)\right)$. Finally, as $w(\cdot)$ is strictly increasing, its monotonicity is equivalent to the monotonicity of $\beta^\eta(\eta,t)$, which, in the end, directly translates into that of $|i(\eta,t)|$.

The second statement can be shown using the quack's indifference between all reports made in equilibrium and the monotonicity of $\beta^m(m,t)$ obtained above. Take $t_1 \in \mathcal{S}$ and assume that $\beta(m,t_1) < \beta_{t_1}$ for both $m \in \{G, B\}$. Then it should be that $\beta(m,t_2) < \beta(m,t_1)$ for both m, since otherwise any report (m,t_2) dominates any report (m,t_1) for the quack – the former grants a higher payoff at t_1, a higher payoff between t_2 and T, and a higher continuation payoff after T. By the

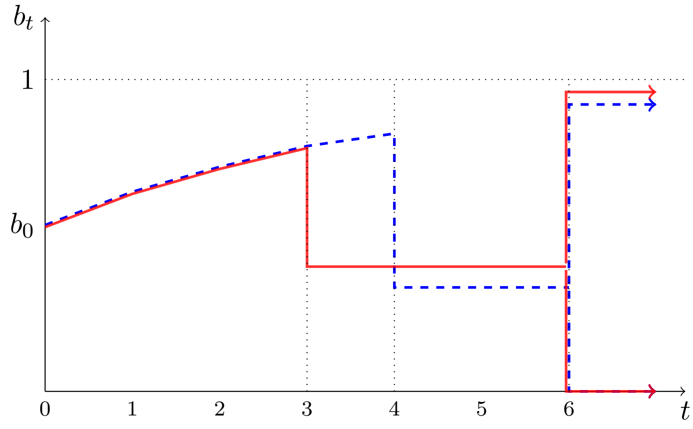

Figure 2: Report penalties increase over time.

martingale property of beliefs, $b(m, t_2)$ and b_{t_2} should average out to b_{t_1}, so in the end we have that

$$\beta(m, t_2) < \beta(m, t_1) < \beta_{t_1} < \beta_{t_2}$$

whenever $\beta(m, t_1) < \beta_{t_1}$. This argument unravels to all $t \in \mathcal{S}$, granting the second and third statements of Theorem 1. This logic is exemplified in Figure 2, where the red solid line shows the reputation path of an analyst who makes a report at $t = 3$, and the blue dashed line shows that for $t = 4$. The logic above does not preclude monotonicity from going the other way if we initially start with inequality $\beta(m, t_1) > \beta_{t_1}$ – but this case would generate a sequence $\beta^m(m, t)$ that is *increasing* in $t \in \mathcal{S}$, which is incompatible with the expert's incentive compatibility discussed in the first part of the theorem.

It is worth noting that as in the example of Section 2, while the inequalities in the statement of Theorem 1 are weak, they can only hold with equality in non-generic cases. The logic behind the result, as presented above, implies that any strict inequalities unravel into further strict inequalities and vice versa, equalities translate into equalities. For example, assuming $\beta(m, t_1) = \beta_{t_1}$ for some $m \in \{G, B\}$ would eventually require (by the quack's indifference) that $\beta(m, t) = \beta_t = \beta_{t_1}$ for all $t \in \mathcal{S}$ and both m; and then once terminal payoffs are taken into account, one can discover that for generic parameter values the resulting system has no solution.

Finally, even the third statement, which is inherently static, requires $|\mathcal{S}| \geqslant 2$. If $|\mathcal{S}| = 1$ (so $\mathcal{S} = \{\bar{t}\}$) then the statement no longer holds: one may construct an equilibrium with $\beta(m, \bar{t}) > \beta_{\bar{t}}$ for both $m \in \{G, B\}$. In such an equilibrium, either report is more likely to be made by an expert than a quack, and so generates a reputation premium. "Static" equilibria (those with $|\mathcal{S}| = 1$) are in this sense potentially more informative than "dynamic" equilibria, and allowing for reports to be made at more than one point in time may actually be harmful to the informativeness of these reports. We discuss this issue in more details in subsection 5.6.

5.5 Existence of Informative Equilibria

So far we have discussed properties of equilibria without proving that any equilibria actually exist, but the existence of informative equilibria is not a trivial concern.[23] The following Proposition

[23] Babbling equilibria, on the other hand, always trivially exist.

outlines some necessary and sufficient conditions for existence of informative equilibria, which allow to understand some driving forces behind their existence and non-existence.

Proposition 4. *Suppose $w(\cdot)$ and $w^c(\cdot)$ are continuous functions and $\pi = 1$. Then*

1. *For any $\bar{t}$ there exists an informative equilibrium with $\mathcal{S} = \{\bar{t}\}$;*

2. *Suppose $w^c(\beta)$ is convex and $p_0 = \frac{1}{2}$. Then for any $\mathcal{S} \subseteq \mathcal{T}$, there exists an informative equilibrium with support $\mathcal{S}$;*[24]

3. *If $w(\beta) = \beta^\alpha$ and $w^c(\beta) = \theta \cdot \beta^\alpha$ with $\theta > 0$ and $\alpha < 1$, then no informative equilibria with $|\mathcal{S}| \geqslant 3$ exist.*

Part 1 of Proposition 4 states that at least some informative equilibria always exist. In particular, there always exist "static" equilibria with singleton support, whatever the single period in the support is. At this period experts reveal all private information they have accumulated by then, and any analyst without private information is also free to make a report in the hopes of guessing the state correctly. However, the main focus of this paper is on the dynamics of announcements, so we are particularly interested in equilibria with $|\mathcal{S}| \geqslant 2$. Part 2 of Proposition 4 gives a sufficient condition for their existence, which is the convexity of the payoff function $w(\cdot)$ and the ex ante symmetry of the two states, $p_0 = \frac{1}{2}$.[25]

Necessary conditions, on the other hand, are not easily obtainable in our model. The reason lies in the fact that the payoff functions $w(\cdot)$ and $w^c(\cdot)$ are only invoked for a finite number of arguments β in any given equilibrium. In particular, given some payoff function $w(\cdot)$ and some equilibrium of the game, we can change values that $w(\cdot)$ and/or $w^c(\cdot)$ take almost everywhere without affecting the equilibrium. This makes necessary conditions difficult to formulate without restricting payoff functions to a specific class, which is what the last part of Proposition 4 does. It states that for at least some class of concave payoff functions the existence of equilibria with large supports ($|\mathcal{S}| \geqslant 3$) completely breaks down.[26] Parts 1 and 3 together illustrate that the main hurdles to existence are tied to intertemporal choice: if the analyst has no choice of *when* to make a report then existence is certain, while allowing predictions to be made at multiple points in time may in some settings lead to complete breakdown of communication.

The reason behind the non-existence is connected to the expert's dynamic incentive compatibility constraints. This is because $t \in \mathcal{S}$ if and only if the informed expert makes a report at t, so he should be willing to do so instead of delaying his report until a later date. This leads to phenomena described in parts 2 and 3 of Theorem 1. In particular, any report has to reduce the analyst's reputation, so the only reason to make the report for the quack is a gamble for the terminal reputation: he should be willing enough to make a guess, understanding that it may be incorrect. A certain degree of risk-loving on behalf of the analyst is required for such a strategy profile to constitute an equilibrium. Conversely, if the quack is too risk-averse then strategy profiles with

[24] If $w^c(\beta)$ is *strictly* convex, the existence of an informative equilibrium for any $\mathcal{S}$ can be guaranteed for any $p_0 \in [\frac{1}{2}, \frac{1}{2} + \varepsilon)$ with $\varepsilon > 0$ depending on the primitives.

[25] An empirical inquiry by Bernhardt, Campello, and Kutsoati [2004] also proposed the convexity of payoffs in reputation as an explanation of the observed dynamics of the analysts' reports. In their case, the observed phenomenon was the strong anti-herding in predictions.

[26] The jump from $|\mathcal{S}| = 1$ to $|\mathcal{S}| \geqslant 3$ is tied to the special features of the Godwin point, which precludes us from making sharp statements about equilibria with $|\mathcal{S}| = 2$. See also Section 6.1.

$|\mathcal{S}| \geqslant 3$ cannot satisfy the incentives of both analyst types at the same time.[27] The formal argument is somewhat more subtle and can be found in the Appendix.

The intuition above naturally leads to the question: do there exist equilibria, given enough risk-aversion on analyst's behalf, in which the quack is too reluctant to make his report for fear of guessing it wrong? Such equilibria do not require sustaining quack's indifference between all reports, so they should seemingly exist under a wider range of parameters and functional forms. Under the current assumptions, the existence of such equilibria violates Proposition 1 and is therefore impossible. However, in Section 6.2 we show that under an alternative assumption on the off-path beliefs such equilibria *can*, in fact, exist, but *only* if $\pi = 1$.

5.6 Comparison of Equilibria

In this section we study how the informativeness of the reports depends on the shape of equilibrium. Simply speaking, we are trying to answer the question of which equilibria are more informative (about the state of the world). We have two characteristics that describe how informative a given equilibrium is: its support $\mathcal{S}$ and two functions $i(m,t)$ for $m \in \{G, B\}$. The support determines how frequently the news are reported, as well as what the deadline $\bar{t}$ for the reports is, and $|i(m,t)|$ shows how noisy a given report is.

As was mentioned in the discussion following Theorem 1, shrinking $\mathcal{S}$ to a singleton can lead to higher informativeness $|i(m,t)|$. In particular, in reticent equilibria, the expert makes a truthful report at or before $\bar{t}$ if and only if he has received a private signal at any $t \leqslant \bar{t}$. In a static equilibrium (one with $\mathcal{S} = \{\bar{t}\}$), we can have $\beta(m,\bar{t}) > \beta_{\bar{t}}$, meaning the quack makes a report at $\bar{t}$ with a smaller probability than the expert, whereas in any equilibrium with $|\mathcal{S}| \geqslant 3$, the converse must hold ($\beta(m,t) < \beta_t$ for all $t \in \mathcal{S}$). Therefore, static equilibria can feature the same amount of signal and less noise than "dynamic" equilibria – but a larger delay on average between the expert receiving a signal and having a chance to report it. In this regard, we conclude that there is a trade-off between the timeliness of information revelation and its quality, at least when comparing equilibria with $|\mathcal{S}| = 1$ and those with $|\mathcal{S}| \geqslant 2$.[28]

Another question is whether extending the reporting deadline $\bar{t}$ to a later date can increase the report informativeness in equilibria with $|\mathcal{S}| \geqslant 2$. The following proposition shows that this is indeed true.

Proposition 5. *Assume that $\pi = 1$ and two reticent informative equilibria exist with the respective supports $\mathcal{S} = \{t_1, ..., t_k\}$, $\tilde{\mathcal{S}} = \{t_1, ..., t_k, t_{k+1}, ..., t_{k+n}\}$ with $k \geqslant 2$ and informativeness measures $i(m,t)$, $\tilde{i}(m,t)$. Then $|i(m,t)| \leqslant |\tilde{i}(m,t)|$ for $m \in \{G, B\}$ and all $t \in \tilde{\mathcal{S}}$.*

The proposition says that expanding support is unambiguously beneficial: it both allows more information to be transmitted by the informed expert (in case he observes his private information between t_k and t_{k+n}) *and* decreases noise of *all* informative reports (weakly for all $t \leqslant t_k$ and strictly for all $t > t_k$). The intuition behind the latter phenomenon is, simply speaking, the more

[27] Remember that $w(\beta_t)$ is a function of $\beta_t = \frac{b_t}{1-b_t}$ which itself is a convex function of b_t. Therefore, even with $\alpha = 1$ the analyst is still risk-loving w.r.t. b_t, so all talks of risk-loving and risk-aversion are in the relative sense (one may easily verify that coefficients of both absolute and relative risk-aversion are monotone in α).

[28] Boleslavsky and Taylor [2020] make a similar point in a substantially different model. Frug [2018] presents a setting where the opposite is true and putting an analyst under time pressure improves the quality of communication due to endogenous learning choices (and no reputational concerns).

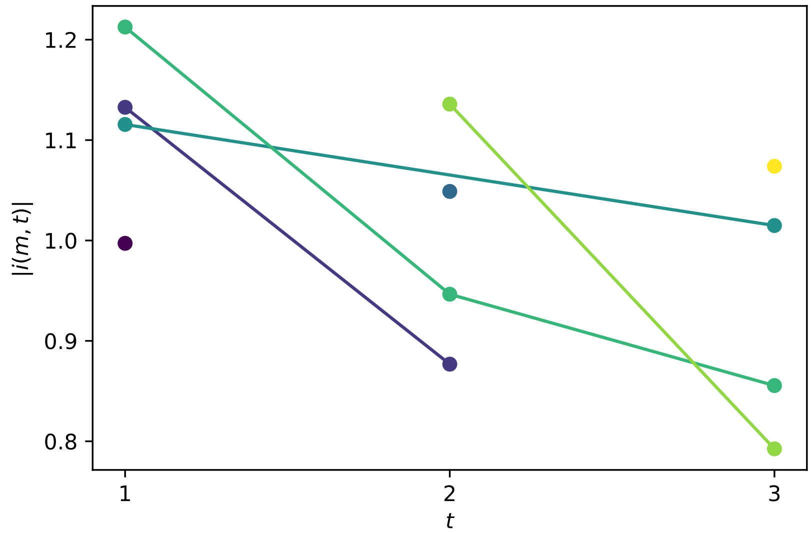

Figure 3: Report informativeness depending on support

Note: the figure assumes $T = 3$, $p_0 = 1/2$, $b_0 = 1/2$, $w(\beta) = \beta^2$, $w^c(\beta) = 100 \cdot w(\beta)$, $F(t) = 1 - 0.5^t$, the seven dots/lines plot message informativeness $|i(m,t)|$ (which coincides over $m \in \{G, B\}$ due to symmetry of the example) over t for all seven informative reticent equilibria with different supports.

reporting options are available to quack in a given equilibrium, the thinner he spreads over them. More specifically, extending the support to the right ceteris paribus implies that the reputation β_t of the silent analyst should improve at the new dates, as per Theorem 1. This renders the option of staying silent more attractive to the quack and does not affect his payoff from making a report. By Proposition 1, the quack should be indifferent between these options, so to restore this indifference after expanding the support we have to make reporting more appealing. This is achieved by prescribing point-wise lower $r^Q(m,t)$ in equilibrium, thereby improving $\beta(m,t)$ and $\beta^\omega(m,t)$ and at the same time depressing β_t.

Figure 3 illustrates Proposition 5, as well as demonstrates the limits to making more general statements. It plots message informativeness $|i(m,t)|$ across periods t in a setting with $T = 3$ for all seven informative reticent equilibria with different supports, corresponding to the seven dots/lines on the Figure. Proposition 5 states that extending message support to the right does, indeed, increase informativeness in all periods in the support. One can infer that from Figure 3 by comparing $|i(m,t)|$ in the equilibrium with $\mathcal{S} = \{1,2,3\}$ to that in the equilibrium with $\mathcal{S} = \{1,2\}$ to that in the equilibrium with $\mathcal{S} = \{1\}$. On the other hand, Figure 3 suggests that no other comparisons can be readily made. Extending message support to the left allows the early news to be communicated in an informative way, but decreases the informativeness of the late messages – one can see that by comparing the equilibria with $\mathcal{S} = \{3\}$ to $\mathcal{S} = \{2,3\}$ and $\mathcal{S} = \{1,2,3\}$. Further, it is not clear whether shifting support to the right is beneficial: the comparison of singleton equilibria ($\mathcal{S} = \{1\}$, $\mathcal{S} = \{2\}$, and $\mathcal{S} = \{3\}$) suggests that delaying the single reporting period is beneficial, since it both increases report informativeness and enables later private signals to be revealed by the experts. However, this comparison fails to extend to non-singleton equilibria: the equilibrium with $\mathcal{S} = \{2,3\}$ has better informativeness in the first reporting period but worse in the second when compared to $\mathcal{S} = \{1, 2\}$. This example suggests that the observer's most preferred equilibrium would depend on how she trades informativeness (signal-to-noise ratio) against the delay and the amount of signals (in the sense that excluding later periods from the support precluds the experts from communicating private signals that are obtained in those periods).

6 Discussion and Extensions

6.1 Informative Equilibria without Reticence

In Theorem 1, we restricted attention to *reticent* equilibria. The main result, however, holds without this restriction, so long as we exclude $\bar{t}$ from parts 2 and 3 of the statement.

Proposition 6. *Fix some $\mathcal{S}$ with $|\mathcal{S}| \geqslant 3$. Suppose that $\pi = 1$ and an informative equilibrium on $\mathcal{S}$ exists. Then in any such equilibrium, the following are true for both $m \in \{G, B\}$:*

1. *later reports are less informative about the state: $|i(m, t)|$ is a decreasing function of t on $\mathcal{S}$;*
2. *the reputation of a silent analyst improves over time: β_t is increasing in t on $\mathcal{S} \backslash \{\bar{t}\}$ and constant on $\mathcal{T} \backslash \mathcal{S}$;*
3. *making any report decreases reputation as compared to no report: $\beta(m, t) \leqslant \beta_t$ for any $t \in \mathcal{S} \backslash \{\bar{t}\}$.*

Proposition 6 differs from Theorem 1 in two respects: it requires $|\mathcal{S}| \geqslant 3$ and excludes $\bar{t}$ from statements 2 and 3. The common reason behind both of these changes is that if we do not restrict attention to reticent equilibria, then the Godwin point $\bar{t}$ differs from other points in $\mathcal{S}$ in that it allows $r^E_\varnothing(m, \bar{t}) > 0$ – that an uninformed expert makes a report, – while from Proposition 2 we know that $r^E_\varnothing(m, t) = 0$ for all $t < \bar{t}$. This can generate situations in which statements 2 and 3 are no longer true at $\bar{t}$, i.e., some report m may have $\beta(m, \bar{t}) > \beta_{\bar{t}}$, while staying silent at $\bar{t}$ would decrease β_t (or be off the equilibrium path altogether). This, however, does not affect the first part of the proposition: the reports made by the uninformed expert at $\bar{t}$ are uninformative, and thus only add more noise, amplifying the effect of decreasing informativeness as compared to reticent equilibria.

6.2 Ideal Equilibria

Informative equilibria with nontrivial supports need not exist with non-convex payoffs, as evidenced by Proposition 4. A question arises: are babbling and small-support equilibria the only possible outcomes when analysts are too risk-averse? The answer is "not necessarily".

The key to answering this question is assumption (OP), which requires that once an analyst has gained perfect reputation, it persists forever – even if his prediction turned out to be wrong. The assumption is known under the name "Never Dissuaded Once Convinced" in the signaling and bargaining literature, see Starkov [2023] for an overview. This assumption can be justified as a limiting case of the model as $\pi \to 1$, i.e., it can be supported by a perturbation of the model in which the expert's signal is incorrect with vanishing probability – and thus so are his predictions

(Section 6.3 explores this setting in more detail and shows that all results hold when $\pi < 1$, under additional conditions).

However, this is not the only possible perturbation of the model in case $\pi = 1$. One may alternatively think of a version of the model with the infinitesimal number of "crazy" analysts who are not strategic in their reports and just voice their opinions at random times, and who may be competent or not. Since their number is infinitesimal, Bayes' rule still prescribes that for any (m,t) such that $r^E_\eta(m,t) > 0 = r^Q(m,t) = r^E_\varnothing(m,t)$ with $\eta = m$, we should have $\beta(m,t) = +\infty$ ($b(m,t) = 1$). However, since an informed expert is never wrong, if such prediction (m,t) turns out incorrect, this would imply that it was actually made by one of the few crazy analysts, who may be competent or not. This perturbation could support any belief $\beta^{-m}(m,t) \in [0, +\infty]$.

In this section we substitute (OP) by an alternative assumption (OP') which prescribes the worst possible off-path belief after an incorrect prediction supposedly made by an expert, same as any other off-path history:

(OP) off the equilibrium path the beliefs are $\rho = \rho_0$ and $\beta = 0$, with the exception that an extreme belief $\beta = +\infty$ ($b = 1$) is not updated;

(OP') off the equilibrium path the beliefs are $\rho = \rho_0$ and $\beta = 0$.

The alternative assumption (OP') allows for the existence of *ideal equilibria*:

Definition. Ideal equilibria *are characterized by* $r^Q(m,t) = r^E_\varnothing(m,t) = 0$ *for all* (m,t), $r^E_\eta(-\eta,t) = 0$ *for all* t *and* η, *and* $r^E_\eta(\eta,t) > 0$ *for some* t *and* η.

Simply speaking, in the ideal equilibria, the only reports ever made are those by the informed experts. The quacks (and the uninformed experts) never voice their opinion, which is enforced by the worst possible terminal reputation if the analyst's report turned out incorrect. For this threat to be sufficient and an ideal equilibrium to exist, the quacks should be sufficiently afraid of the chance of getting stuck with bad reputation in the long run, relative to the gains from good reputation in the short term. In other words, the payoff from reputation $w(\cdot)$ must be sufficiently concave. While the exact condition is more subtle than the concavity of $w(\cdot)$ and $w^c(\cdot)$, we can show the converse: if $w(\cdot)$ or $w^c(\cdot)$ are convex then ideal equilibria do not exist. This result is the exact opposite of part 2 of Proposition 4, meaning that ideal equilibria are, informally speaking, complementary to informative equilibria in the sense of existence.

Proposition 7. *Suppose* $\pi = 1$. *If either* $w(\cdot)$, *or* $w^c(\cdot)$ *is convex, then there exists no ideal equilibrium that satisfies (OP'), (SY), and (ML).*

For an interested reader, the proof in the Appendix states the exact necessary and sufficient condition for the existence of an ideal equilibrium with a given support $\mathcal{S}$.

6.3 Imperfect Private Signals

In this section we relax the assumption that the expert's signals are perfectly informative about the state and explore the case $\pi < 1$. Note that there is nothing in the intuition behind Theorem 1 implying that $\pi = 1$ is necessary. As long as $\pi > \frac{1}{2}$, the expert's signal is somewhat informative about the state, so his informed report about the state is more likely to be supported by the ex post evidence than the quack's random guess.

Propositions 1, 2, 3, and 7 continue to hold in case $\pi < 1$ with no further modifications.[29] Proposition 8 below shows that the remaining results continue to hold as well if $w^c(\cdot)$ is either convex, or at least not too globally concave, and the private signal is sufficiently precise (on top of the requirements imposed in the respective statements).

Proposition 8. *Theorem 1 and Propositions 4– 6 are true for $\pi < 1$ if either of the following holds:*

1. *$w^c(\cdot)$ is convex;*
2. *$w^c(\cdot)$ is continuously differentiable and there exist $0 < \underline{d} \leqslant \bar{d} < +\infty$ such that $\frac{dw^c(\beta)}{d\beta} \in [\underline{d}, \bar{d}]$ for all β, and $\pi > \frac{\bar{d}}{\underline{d}+\bar{d}}$.*

We note that either of these conditions imply that the terminal payoff of (surely) having the expert reputation has to be infinite, and convex at the top. This aligns well with many expert markets being superstar markets, where the top tail of analysts that are [perceived to be] the most competent attract a disproportionately large part of the business. Li [2007] establishes the convexity of payoff to reputation in the theoretical model. Groysberg, Healy, and Maber [2011] show that this holds in financial analyst markets: all else equal, the "All-Star" recognition produces a premium to the analysts' compensation.[30] In another application, Heckman and Moktan [2020] show that researchers from top Economics departments have easier access to publication in top-five Economics journals, which, in turn, significantly improves their career prospects, so the payoff from initial reputation is convex in the academic job market. Finally, Li and Martin [2019] demonstrate that for entrepreneurs on Kickstarter, the effect of positive past reputation on fundraising success is stronger than that of negative past reputation.

The intuition behind Proposition 8 can be seen as follows. When describing the intuition behind Theorem 1, we have mentioned that in order to provide the incentives for the informed expert to reveal his private information immediately instead of waiting for a later date, the premium $\Delta w_\eta(m,t)$ for guessing the state correctly, as given by (11), should be decreasing in t on $\mathcal{S}$. An important part of the proof of Theorem 1 consists of showing that decreasing $\Delta w_\eta(m,t)$ is equivalent to decreasing $\beta^m(m,t)$. This equivalence is immediate when $\pi = 1$, since then $\beta^{-m}(m,t) = 0$, and $w^c(\cdot)$ is a strictly increasing function. Proposition 8 provides two alternative conditions under which the equivalence holds in case $\pi < 1$. If $w^c(\cdot)$ is convex, it holds due to $b^m(m,t)$ and $b^{-m}(m,t)$ being scalar multiples of each other. The second condition relaxes convexity to just bounded derivative of $w^c(\cdot)$ but the idea is the same: if $\frac{dw^c(\beta)}{d\beta}$ is bounded so that $w^c(\cdot)$ is not too concave globally, and the signal is precise enough, we can establish the desired connection between $\Delta w_\eta(m,t)$ and $\beta^m(m,t)$.

It is also worth noting that ideal equilibria outlined in Section 6.2 can no longer exist if $\pi \in \left(\frac{1}{2}, 1\right)$. This is because the analyst who is believed competent with probability one can no longer be punished after his prediction was revealed to be wrong – he can credibly claim that the mistake was made because of an incorrect private signal, rather than due to low competence.

[29]Proposition 7 can be extended to claim that no ideal equilibria exist regardless of $w(\cdot)$ and $w^c(\cdot)$. This is because assumption (OP') is equivalent to (OP) when $\pi < 1$.

[30][Yin and Zhang, 2014, p. 574] claim that "this seems to be a widely accepted fact: the popular press suggests that a large pay discrepancy exists between All-Star analysts and Non-All-Star analysts."

6.4 Other Information Measures

Our measure of message informativeness $i(m,t)$ as given by (10) is intuitive, yet ad hoc, and is not commonly used in the literature. In this section, we replace it with a more standard measure of "distance" between two probability distributions (beliefs in our case), the Kullback-Leibler divergence, also known as the relative entropy. We show that all of our results continue to hold when KL divergence is used to measure message informativeness instead of $|i(m,t)|$.

Given two discrete probability distributions $p, q \in \Delta(\Omega)$, KL divergence of p from q is defined as

$$D_{KL}(p||q) := \sum_{\omega\in\Omega} p_\omega \cdot \ln\left(\frac{p_\omega}{q_\omega}\right),$$

KL divergence of p from q can be interpreted as the information gain of p over q when p is the true distribution. For our purposes it also helps to note that $D_{KL}(p||q) \geqslant 0$, with equality if and only if $p = q$. See Cover and Thomas [2006] for more details. Our goal is to measure the information gain produced by a given report (m,t), which moves the public belief from p_{t-1} to $p(m,t)$. It is then reasonable to define the informativeness of (m,t) as

$$i_{KL}(m,t) := D_{KL}\left(p(m,t)||p_{t-1}\right) = p(m,t)\cdot\ln\left(\frac{p(m,t)}{p_{t-1}}\right) + (1-p(m,t))\cdot\ln\left(\frac{1-p(m,t)}{1-p_{t-1}}\right).$$

Note that $i_{KL}(m,t)$ is not simply a monotone transformation of $i(m,t)$. The following proposition shows, however, that our results hold with either informativeness measure.

Proposition 9. *Theorem 1 and Propositions 5, 6, and 8 continue to hold if $i(m,t)$ is replaced by $i_{KL}(m,t)$ in all statements.*

Only the results mentioned in Proposition 9 make statements regarding $i(m,t)$, meaning that all results survive under $i_{KL}(m,t)$. Note, in particular, that the definitions of a babbling report, an informative report, and an informative equilibrium are independent of the informativeness measure employed.

6.5 Commitment

Suppose now that the analyst can commit to a reporting strategy at $t = 0$ after learning his type but before receiving any private information. The analyst's strategy is not publicly observable, so he cannot use it to signal his type. This modification relates our problem to the literature on Bayesian Persuasion and information design (see Bergemann and Morris [2019] for a recent survey), since the analyst now designs the disclosure strategy subject to the constraints on the information available to him.

The literature on Bayesian Persuasion has demonstrated that commitment power often allows the sender to strictly improve his payoff whenever the optimal communication mechanism is informative (c.f. Lipnowski, Ravid, and Shishkin [2022]). In contrast, it is easy to see that in our setting, all of the analyst's strategy profiles that were optimal in the absence of commitment remain optimal even if he has commitment power. In particular, the analyst's problem features no dynamic inconsistency, since it is always in his best interest to provide as correct a report as possible. The quack receives no information throughout the game, hence always remains indifferent between all the reports sent in equilibrium and staying silent, and (OP) implies that the off-path actions are no better. Therefore, conditional on the expert's strategy, the quack cannot improve by committing to a strategy that is different from what is prescribed by equilibrium. On the other hand, the expert's incentives to provide the correct report imply that it is optimal to send the correct report once the private signal arrives (and Theorem 1 implies it is optimal to do so as soon as possible), and silently wait for the signal otherwise. Therefore, the analyst's commitment power does not affect the equilibria identified above.

7 Conclusion

The paper presents a model of dynamic cheap talk in the presence of career concerns. We discover that competition between competent (experts) and incompetent (quacks) analysts imposes plenty of structure on equilibrium outcomes. In particular, we show that to incentivize the experts – whose reports drive the whole market, – to make early predictions, it must be that early reports are perceived more favorably by the public than later reports. Perhaps more surprisingly, we discover that the presence of quacks in the market together with the monotonicity above generates an automatic penalty for any report: an analyst who makes a prediction will see his reputation plummeting, and he will only be redeemed if his prediction will turn out to be correct. This does not discourage the quacks from speaking up, but disciplines their incentives. Moreover, this reputation dynamics implies that for non-trivial equilibria to exist, analysts' payoffs must be sufficiently convex in reputation, which is the case if, e.g., the premium for being the top analyst in the field is very large.

These predictions are novel in the literature, and are driven by us explicitly modeling the dynamic payoff structure of the analysts. Our model accounts for both flow payoffs while the public is still uncertain about the correctness of the analyst's prediction, and terminal payoffs realized after the true state is revealed. The model can be extended in multiple directions, e.g., to account for competition among analysts, or for the arrival of public signal in the background. Richer private news processes for analysts can also add another strategic layer to the timing decision of the analyst's prediction. All of these are prospective avenues for future research.

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Appendix

For the proofs, it is useful to introduce a shorthand notation for the analyst's expected payoff from making report m in period $\tau \geqslant t$ at history h_t^γ:

$$W_{t,\eta}^\gamma(m,\tau) := \mathbb{E}\left[\sum_{s=t}^{T-1} w\left(\beta(h_s^p)\right) + w^c\left(\beta(h_T^p)\right)\middle| t, \eta, \mu_\tau = (m,\tau)\right]. \tag{12}$$

With this notation we have that report (m,t) is optimal at t if and only if $V_{t,\eta}^\gamma = W_{t,\eta}^\gamma(m,t)$. Moreover, we use $W_{t,\eta}^\gamma(\varnothing)$ to denote the respective value from not making any report until the end of period T (i.e., conditional on $\mu_T = \varnothing$). For the quack we omit subscript η and use $W_t^Q(m,\tau)$ and $W_t^Q(\varnothing)$, because the quack never gets any private information.

Proof of Proposition 1. The proof is valid for all $\pi \in \left(\frac{1}{2}, 1\right]$. We show that $r^E(m,t) > 0$ if and only if $r^Q(m,t) > 0$ for any (m,t) for any history with $b(h_t^p) \in (0,1)$. Together with the fact that $b_0 \in (0,1)$, this will then mean that on equilibrium path we never arrive at a [non-terminal] history with $b(h_t^p) \in \{0,1\}$, hence the statement is true for all histories on equilibrium path.

Part 1: $r^E(m,t) > 0 \Rightarrow r^Q(m,t) > 0$. Suppose by contradiction that $r^Q(m,t) = 0$. Then $b(m,t) = 1$ and it does not change afterwards due to (OP), meaning that $W_t^Q(m,t)$ attains maximum among all continuation payoffs (feasible or not). The initial assumption $r^Q(m,t) = 0$ then means that either $W_t^Q(\varnothing)$, or $W_t^Q(m,s)$ for some m and $s > t$ attain maximum, since one of these options should be at least as good as report (m,t). These payoffs, however, cannot attain maximum, since $b_t < 1$.

Part 2: $r^Q(m,t) > 0 \Rightarrow r^E(m,t) > 0$. Suppose that $r^E(m,t) = 0$. Since $r^Q(m,t) > 0$, we have $b(m,t) = 0$, and it does not change afterwards. Hence $W_t^Q(m,t)$ attains minimum among all continuation payoffs. However, since belief about the analyst's type is a martingale from the observer's point of view, we have either $b(-m,t) > 0$ or $b_{t+1} > 0$. Thus, at least one of these strategies (reporting $-m$ or staying silent at t) strictly dominates the strategy of reporting (m,t) for the quack, so $r^Q(m,t) = 0$.

Similarly, one can show that $r^Q(G,s) + r^Q(B,s) = 1$ if and only if $r^E(G,s) + r^E(B,s) = 1$. Indeed, if for some t we have $r^E(G,t) + r^E(B,t) = 1$ and $r^Q(G,t) + r^Q(B,t) < 1$, then not making a report by t grants the quack a continuation payoff of zero, while by the martingale property of belief there exists $m \in \{G,B\}$ such that $b(m,t) > 0$, and therefore report (m,t) dominates the strategy of staying silent. Similarly, if $r^E(G,t) + r^E(B,t) < 1$ and $r^Q(G,t) + r^Q(B,t) = 1$, then not making a report by t yields the maximal continuation payoff, while again by the martingale property making at least some report gives strictly less in expectation. □

Before we proceed, it is useful to introduce some new pieces of notation which will come in handy for further proofs. The expert's report probabilities can be rewritten as

$$\begin{aligned} r^E(m,t) = \mathbb{E}_\eta\left[r_\eta^E(m,t)\right] &= \frac{p_t^* \cdot z_{t,G} \cdot r_G^E(m,t) + (1-p_t^*) \cdot z_{t,B} \cdot r_B^E(m,t) + z_{t,\varnothing} \cdot r_\varnothing^E(m,t)}{p_t^* \cdot z_{t,G} + (1-p_t^*) \cdot z_{t,B} + z_{t,\varnothing}}, \\ \mathbb{E}_\eta[r_\eta^E(m,t)|\omega] &= \frac{\pi \cdot z_{t,\omega} \cdot r_\omega^E(m,t) + (1-\pi) \cdot z_{t,-\omega} \cdot r_{-\omega}^E(m,t) + z_{t,\varnothing} \cdot r_\varnothing^E(m,t)}{\pi \cdot z_{t,\omega} + (1-\pi) \cdot z_{t,-\omega} + z_{t,\varnothing}}, \end{aligned} \tag{13}$$

where $p_t^* := p_t \cdot \pi + (1-p_t) \cdot (1-\pi)$, and $z_{t,\eta} := P\left\{\eta_t = \eta, \mu_{t-1} = \varnothing | \eta_{t^*}^E = \eta\right\}$ for $\eta \in \{\varnothing, G, B\}$. In words, $z_{t,\eta}$ is the probability that the expert has information η at time t and has not made a report prior to t, conditional on expert's signal realization being $\eta_{t^*}^E = \eta$ (or unconditional if $\eta = \varnothing$). It can be expressed recursively as

$$
\begin{aligned}
z_{t,\eta} &= z_{t-1,\eta} \cdot \left(1 - \sum_m r^E_\eta(m, t-1)\right) + z_{t-1,\varnothing} \cdot \lambda(t) \cdot \left(1 - \sum_m r^E_\varnothing(m, t-1)\right), \\
z_{t,\varnothing} &= z_{t-1,\varnothing} \cdot (1 - \lambda(t)) \cdot \left(1 - \sum_m r^E_\varnothing(m, t-1)\right),
\end{aligned} \tag{14}
$$

with $z_{0,G} = z_{0,B} = 0$ and $z_{0,\varnothing} = 1$.

Next, following report (m, t), the observer's belief about the state is updated as:

$$
\rho(m,t) = \rho_{t-1} \cdot \frac{(1-b_{t-1}) \cdot r^Q(m,t) + b_{t-1} \cdot \mathbb{E}_\eta\left[r^E_\eta(m,t)|\omega = G\right]}{(1-b_{t-1}) \cdot r^Q(m,t) + b_{t-1} \cdot \mathbb{E}_\eta\left[r^E_\eta(m,t)|\omega = B\right]} = \rho_{t-1} \cdot \frac{1 + \beta^G(m,t)}{1 + \beta^B(m,t)}. \tag{15}
$$

This shows that $\rho(m, t) = \rho_{t-1}$ if and only if $\beta^G(m, t) = \beta^B(m, t)$.

To shorten the proof of Proposition 2 we next establish two auxiliary lemmas, both of which are valid for all $\pi \in (\frac{1}{2}, 1]$.

Lemma 10. *Fix any equilibrium and period t such that reports (G, t) and (B, t) are both on equilibrium path. If the expert with private signal $\eta = G$ (strictly) prefers report $m' \in \{G, B\}$ over $m'' \in \{G, B\}$ at t, then the expert with private signal $\eta = B$ (strictly) prefers m'' over m', and vice versa*

Proof. If both (G, t) and (B, t) are on path, then the quack should be indifferent between the two reports: $W^Q_t(G, t) = W^Q_t(B, t)$, meaning

$$
\begin{aligned}
&(T-t) \cdot w(\beta(G,t)) + p_0 \cdot w^c(\beta^G(G,t)) + (1-p_0) \cdot w^c(\beta^B(G,t)) = \\
&\qquad = (T-t) \cdot w(\beta(B,t)) + p_0 \cdot w^c(\beta^G(B,t)) + (1-p_0) \cdot w^c(\beta^B(B,t)).
\end{aligned} \tag{16}
$$

As assumed, the expert with $\eta = G$ prefers report (m', t) to (m'', t): $W^E_{t,G}(m', t) \geqslant W^E_{t,G}(m'', t)$, or

$$
\begin{aligned}
&(T-t) \cdot w(\beta(m',t)) + \frac{1}{p_0^*} \cdot \Big[p_0 \cdot \pi \cdot w^c(\beta^G(m',t)) + (1-p_0) \cdot (1-\pi) \cdot w^c(\beta^B(m',t))\Big] \geqslant \\
&\qquad \geqslant (T-t) \cdot w(\beta(m'',t)) + \frac{1}{p_0^*} \cdot \Big[p_0 \cdot \pi \cdot w^c(\beta^G(m'',t)) + (1-p_0) \cdot (1-\pi) \cdot w^c(\beta^B(m'',t))\Big].
\end{aligned}
$$

Multiplying the inequality above by p_0^*, subtracting it from (16), and dividing the result by $(1 - p_0^*)$, we get

$$
\begin{aligned}
&(T-t) \cdot w(\beta(m',t)) + \frac{1}{1-p_0^*} \cdot \Big[p_0 \cdot (1-\pi) \cdot w^c(\beta^G(m',t)) + (1-p_0) \cdot \pi \cdot w^c(\beta^B(m',t))\Big] \leqslant \\
&\qquad \leqslant (T-t) \cdot w(\beta(m'',t)) + \frac{1}{1-p_0^*} \cdot \Big[p_0 \cdot (1-\pi) \cdot w^c(\beta^G(m'',t)) + (1-p_0) \cdot \pi \cdot w^c(\beta^B(m'',t))\Big],
\end{aligned}
$$

which exactly means that expert with $\eta = B$ prefers report (m'', t) over (m', t). Further, this preference is strict if and only if the preference of the expert with $\eta = G$ is strict. $\square$

Lemma 11. *Fix any equilibrium and period t such that reports (G, t) and (B, t) are both on equilibrium path. If the expert with private signal $\eta \in \{G, B\}$ is indifferent between reports (G, t) and (B, t), then both reports must induce the same observer's beliefs: $\beta(G, t) = \beta(B, t)$ and $\beta^\omega(G, t) = \beta^\omega(B, t)$ for both $\omega \in \{G, B\}$.*

Proof. If both reports (G, t) and (B, t) are on path, the quack should be indifferent between these reports, as captured by condition (16). In turn, the indifference of the expert with information $\eta = G$ (case $\eta = B$

is analogous) is represented by the following condition: $W^E_{t,G}(G,t) = W^E_{t,G}(B,t)$ or, equivalently,

$$(T-t)\cdot w(\beta(G,t)) + \frac{1}{p_0^*}\cdot\Big[p_0\cdot\pi\cdot w^c(\beta^G(G,t)) + (1-p_0)\cdot(1-\pi)\cdot w^c(\beta^B(G,t))\Big] =$$
$$= (T-t)\cdot w(\beta(B,t)) + \frac{1}{p_0^*}\cdot\Big[p_0\cdot\pi\cdot w^c(\beta^G(B,t)) + (1-p_0)\cdot(1-\pi)\cdot w^c(\beta^B(B,t))\Big].$$

Subtracting this equality from (16) and rearranging, we get

$$w^c\left(\beta^G(G,t)\right) - w^c\left(\beta^B(G,t)\right) = w^c\left(\beta^G(B,t)\right) - w^c\left(\beta^B(B,t)\right).$$

Therefore, if $\beta^G(G,t) \geqslant \beta^G(B,t)$, then it must be that $\beta^B(G,t) \geqslant \beta^B(B,t)$, and if $\beta^G(G,t) \leqslant \beta^G(B,t)$, then $\beta^B(G,t) \leqslant \beta^B(B,t)$.

At the same time, (6) and (13) together imply that

$$\beta(m,t) = y_t\cdot\beta^G(m,t) + (1-y_t)\cdot\beta^B(m,t), \tag{17}$$

where

$$y_t = \frac{p_t\cdot(\pi\cdot z_{t,G} + (1-\pi)\cdot z_{t,B} + z_{t,\varnothing})}{p_t^*\cdot z_{t,G} + (1-p_t^*)\cdot z_{t,B} + z_{t,\varnothing}}.$$

Representation (17) implies that if $\beta^G(G,t) \geqslant \beta^G(B,t)$ and $\beta^B(G,t) \geqslant \beta^B(B,t)$, then $\beta(G,t) \geqslant \beta(B,t)$, meaning the quack's indifference (16) can be sustained only if $\beta^G(G,t) = \beta^G(B,t)$, $\beta^B(G,t) = \beta^B(B,t)$, and $\beta(G,t) = \beta(B,t)$. The other case – $\beta^G(G,t) \leqslant \beta^G(B,t)$ and $\beta^B(G,t) \leqslant \beta^B(B,t)$ – leads to the same conclusion. □

Proof of Proposition 2. The proof is valid for all $\pi \in (\frac{1}{2}, 1]$. Denote the points of support $\mathcal{S} = \{t_1, t_2, \dots, t_{|\mathcal{S}|}\}$.

First, the existence of a Godwin point as defined by $\bar{t} := \min\{t \in \mathcal{T} \mid V^E_{t,\varnothing} = V^Q_t\}$ is trivial, since the required equality is always satisfied for $t_{|\mathcal{S}|}$, the last point in $\mathcal{S}$. To see this, observe that any report (m,t) for $t > t_{|\mathcal{S}|}$ yields zero reputation for the rest of the game due to assumption (OP), and is therefore weakly dominated for any type of the analyst by staying silent. At the same time, staying silent or reporting any state at $t_{|\mathcal{S}|}$ yields the same expected payoff to the uninformed expert as it does to the quack, since they have the same information. This gives the result. The remainder of this proof demonstrates that $\bar{t}$ satisfies the other requirements stated in the proposition.

Step 1. We next consider $t_{|\mathcal{S}|}$ and show that either all reports are babbling at $t_{|\mathcal{S}|}$, or $r^E_G(G, t_{|\mathcal{S}|}) = r^E_B(B, t_{|\mathcal{S}|}) = 1$. The quack is indifferent between making either report and staying silent if the latter is on path at $t_{|\mathcal{S}|}$. Next, the uninformed expert is indifferent between all these options as well: he neither knows the state at $t_{|\mathcal{S}|}$, nor has an option to communicate it in further periods, as $t_{|\mathcal{S}|}$ is the last point of the support. Note that at $t_{|\mathcal{S}|}$, staying silent can be interpreted as a report, because the analyst can not make any reports beyond $t_{|\mathcal{S}|}$. Therefore, Lemmas 10 and 11 at $t_{|\mathcal{S}|}$ can be applied not only to reports, but also to the option of staying silent.

Note also that $t_{|\mathcal{S}|} \in \mathcal{S}$ implies at least one report $m \in \{G, B\}$ is on the equilibrium path at $t_{|\mathcal{S}|}$. If exactly one report m is on path and not reporting is off path, then $(m, t_{|\mathcal{S}|})$ is trivially babbling.

If not reporting is on path, (SY) and (ML) together imply that if $t_{|\mathcal{S}|} \in \mathcal{S}_m \backslash \mathcal{S}_{-m}$ for some m, then $r^E_\eta\left(m, t_{|\mathcal{S}|}\right) = 0$ for $\eta \in \{G, B\}$. So by (15) and (13), $\rho\left(m, t_{|\mathcal{S}|}\right) = \rho_{t_{|\mathcal{S}|-1}}$. Since $-m$ is off-path, this implies that $\rho_{t_{|\mathcal{S}|}} = \rho_{t_{|\mathcal{S}|-1}}$. From $r^E_\eta\left(m, t_{|\mathcal{S}|}\right) = 0$ for both $\eta \in \{G, B\}$, it follows that the informed expert at least weakly prefers $\varnothing$ over m at $t_{|\mathcal{S}|}$. Therefore, by Lemma 10, the informed expert should be indifferent between staying silent and report m for any $\eta \in \{G, B\}$. Consequently, Lemma 11 implies that $\beta(m,t) = \beta_{t_{|\mathcal{S}|}}$ and, since $-m$ is off-path, that $\beta(m,t) = \beta_{t_{|\mathcal{S}|-1}}$. We thus conclude that $(m, t_{|\mathcal{S}|})$ is indeed a babbling message.

Hence it only remains to show the statement of this step for the case when both reports $m \in \{G, B\}$ are on the equilibrium path at $t_{|\mathcal{S}|}$. We proceed to show it case-by-case, for the exhaustive set of cases according to the preference of the informed expert with $\eta = G$. Note that if the expert strictly prefers to stay silent in the respective case, then (OP) immediately implies that doing so is on the equilibrium path.

Case 1: The expert with $\eta = G$ strictly prefers to make report G over other options. Then $r^E_G(G, t_{|\mathcal{S}|}) = 1$, and by (SY) we get $r^E_B(B, t_{|\mathcal{S}|}) = 1$, which supports the claim.

Case 2: The expert with $\eta = G$ strictly prefers to make report B over other options. Then $r^E_G(B, t_{|\mathcal{S}|}) = 1$ and by (SY) we get $r^E_G(G, t_{|\mathcal{S}|}) = r^E_B(B, t_{|\mathcal{S}|}) = 0$. Since $r^E_B(G, t_{|\mathcal{S}|}) \geqslant 0$, this then violates (ML).

Case 3: The expert with $\eta = G$ strictly prefers to stay silent over other options. Then $r^E_G(G, t_{|\mathcal{S}|}) = r^E_G(B, t_{|\mathcal{S}|}) = 0$ and by (SY), $r^E_B(B, t_{|\mathcal{S}|}) = 0$. By Lemma 10, the expert with $\eta = B$ must strictly prefer to report either B, or G over staying silent. Therefore, $r^E_B(G, t_{|\mathcal{S}|}) = 1$ which, in turn, violates (ML).

Case 4: The expert with $\eta = G$ is indifferent between reporting G and B, and strictly prefers these options to staying silent, which is on the equilibrium path. If the latter is on path, then by Lemma 10, the expert with $\eta = B$ strictly prefers to stay silent. Then $r^E_B(G, t_{|\mathcal{S}|}) = r^E_B(B, t_{|\mathcal{S}|}) = 0$. By (SY) we get $r^E_G(G, t_{|\mathcal{S}|}) = 0$, so $r^E_G(B, t_{|\mathcal{S}|}) = 1$, which then violates (ML).

Case 5: The expert with $\eta = G$ is indifferent between reporting G and staying silent and strictly prefers these options to reporting B. Then by Lemma 10, the expert with $\eta = B$ strictly prefers to report B. Then $r^E_B(B, t_{|\mathcal{S}|}) = 1$, and by (SY) we get $r^E_G(G, t_{|\mathcal{S}|}) = 1$, which supports the claim.

Case 6: The expert with $\eta = G$ is indifferent between reporting B and staying silent and strictly prefers these options to reporting G. Then by (SY) we get $r^E_G(G, t_{|\mathcal{S}|}) = r^E_B(B, t_{|\mathcal{S}|}) = 0$. By Lemma 10, the expert with $\eta = B$ strictly prefers to report G, and therefore $r^E_B(G, t_{|\mathcal{S}|}) = 1$ and $r^E_B(B, t_{|\mathcal{S}|}) = 0$, which, in turn, violates (ML) because $r^E_G(B, t_{|\mathcal{S}|}) \geqslant 0$.

Case 7: The expert with $\eta = G$ is indifferent between all $m \in \{G, B\}$ and staying silent if the latter is on the equilibrium path.[31] By Lemma 10, the expert with $\eta = B$ is indifferent between the same options. By Lemma 11, all options must induce the same beliefs: $\beta^G(G, t_{|\mathcal{S}|}) = \beta^G(B, t_{|\mathcal{S}|}) = \beta^G(\varnothing)$ and $\beta^B(G, t_{|\mathcal{S}|}) = \beta^B(B, t_{|\mathcal{S}|}) = \beta^B(\varnothing)$ (with the latter equality in each pair only being relevant if $m = \varnothing$ is on path at $t_{|\mathcal{S}|}$). We next show that these two groups of beliefs should be equal to each other. Suppose not and $\beta^G(m, t_{|\mathcal{S}|}) > \beta^B(m, t_{|\mathcal{S}|})$ for $m \in \{G, B\}$ and $\beta^G(\varnothing) > \beta^B(\varnothing)$ (the other case – when $\beta^G(m, t_{|\mathcal{S}|}) < \beta^B(m, t_{|\mathcal{S}|})$ for $m \in \{G, B\}$ and $\beta^G(\varnothing) < \beta^B(\varnothing)$ – is analogous). Given (13), these inequalities amount to

$$\frac{\pi z_{t,G} r^E_G(m,t) + (1-\pi) z_{t,B} r^E_B(m,t) + z_{t,\varnothing} r^E_\varnothing(m,t)}{\pi z_{t,G} + (1-\pi) z_{t,B} + z_{t,\varnothing}} > \frac{\pi z_{t,B} r^E_B(m,t) + (1-\pi) z_{t,G} r^E_G(m,t) + z_{t,\varnothing} r^E_\varnothing(m,t)}{\pi z_{t,B} + (1-\pi) z_{t,G} + z_{t,\varnothing}}$$

for all $m \in \{G, B, \varnothing\}$. Here with some abuse of notation we let $r^E_\eta(\varnothing, t) := 1 - r^E_\eta(G, t) - r^E_\eta(B, t)$. However, if we add up all the inequalities above for $m \in \{G, B, \varnothing\}$, we get that $1 > 1$, which is a clear contradiction.

[31] If $m = \varnothing$ is off path, the argument still applies so long as one omits all beliefs related to making no report.

Therefore, $\beta^G(m, t_{|\mathcal{S}|}) = \beta^B(m, t_{|\mathcal{S}|}) = \beta(m, t_{|\mathcal{S}|})$ for $m \in \{G, B\}$ and $\beta^G(\varnothing) = \beta^B(\varnothing) = \beta_{t_{|\mathcal{S}|}}$. Given (15), we then have $\rho(m, t_{|\mathcal{S}|}) = \rho_{t_{|\mathcal{S}|-1}}$, which establishes one condition for $\left(m, t_{|\mathcal{S}|}\right)$ to be a babbling report. Moreover, given (17), we have that revelation of state does not affect the analyst's reputation irrespective of whether he made a report at $t_{|\mathcal{S}|}$ or not. Then due to the martingale property of beliefs, for the expert to be indifferent between either report and staying silent, it must be that

$$\beta(G, t_{|\mathcal{S}|}) = \beta(B, t_{|\mathcal{S}|}) = \beta_{t_{|\mathcal{S}|}} = \beta_{t_{|\mathcal{S}|-1}},$$

which concludes the proof that either report is babbling at $t_{|\mathcal{S}|}$.

Step 2. We next show that if $r^E_G(G, t_{|\mathcal{S}|}) = r^E_B(B, t_{|\mathcal{S}|}) = 1$ then $V^E_{t_{|\mathcal{S}|},\eta} > V^Q_{t_{|\mathcal{S}|}}$ for $\eta \in \{G, B\}$. Since both $m \in \{G, B\}$ are optimal for the quack at $t_{|\mathcal{S}|}$, we have

$$\begin{aligned} V^E_{t_{|\mathcal{S}|},G} - V^Q_{t_{|\mathcal{S}|}} &= W^E_{t_{|\mathcal{S}|},G}(G, t_{|\mathcal{S}|}) - W^Q_{t_{|\mathcal{S}|}}(G, t_{|\mathcal{S}|}) = \\ &= \frac{(2\pi - 1) \cdot p_0 \cdot (1 - p_0)}{p^*_0} \cdot \left(w^c(\beta^G(G, t_{|\mathcal{S}|})) - w^c(\beta^B(G, t_{|\mathcal{S}|}))\right), \\ V^E_{t_{|\mathcal{S}|},B} - V^Q_{t_{|\mathcal{S}|}} &= W^E_{t_{|\mathcal{S}|},B}(B, t_{|\mathcal{S}|}) - W^Q_{t_{|\mathcal{S}|}}(B, t_{|\mathcal{S}|}) = \\ &= \frac{(2\pi - 1) \cdot p_0 \cdot (1 - p_0)}{1 - p^*_0} \cdot \left(w^c(\beta^B(B, t_{|\mathcal{S}|})) - w^c(\beta^G(B, t_{|\mathcal{S}|}))\right). \end{aligned} \tag{18}$$

Therefore, to establish the claim we need to verify that $\beta^G(G, t_{|\mathcal{S}|}) > \beta^B(G, t_{|\mathcal{S}|})$ and $\beta^B(B, t_{|\mathcal{S}|}) > \beta^G(B, t_{|\mathcal{S}|})$. Given $r^E_G(G, t_{|\mathcal{S}|}) = r^E_B(B, t_{|\mathcal{S}|}) = 1$ and (13) these inequalities are equivalent to

$$\begin{aligned} \frac{\pi \cdot z_{t_{|\mathcal{S}|},G} + z_{t_{|\mathcal{S}|},\varnothing} \cdot r^E_\varnothing(G, t_{|\mathcal{S}|})}{\pi \cdot z_{t_{|\mathcal{S}|},G} + (1 - \pi) \cdot z_{t_{|\mathcal{S}|},B} + z_{t_{|\mathcal{S}|},\varnothing}} &> \frac{(1 - \pi) \cdot z_{t_{|\mathcal{S}|},G} + z_{t_{|\mathcal{S}|},\varnothing} \cdot r^E_\varnothing(G, t_{|\mathcal{S}|})}{\pi \cdot z_{t_{|\mathcal{S}|},B} + (1 - \pi) \cdot z_{t_{|\mathcal{S}|},G} + z_{t_{|\mathcal{S}|},\varnothing}}, \\ \frac{\pi \cdot z_{t_{|\mathcal{S}|},B} + z_{t_{|\mathcal{S}|},\varnothing} \cdot r^E_\varnothing(B, t_{|\mathcal{S}|})}{\pi \cdot z_{t_{|\mathcal{S}|},B} + (1 - \pi) \cdot z_{t_{|\mathcal{S}|},G} + z_{t_{|\mathcal{S}|},\varnothing}} &> \frac{(1 - \pi) \cdot z_{t_{|\mathcal{S}|},B} + z_{t_{|\mathcal{S}|},\varnothing} \cdot r^E_\varnothing(B, t_{|\mathcal{S}|})}{\pi \cdot z_{t_{|\mathcal{S}|},G} + (1 - \pi) \cdot z_{t_{|\mathcal{S}|},B} + z_{t_{|\mathcal{S}|},\varnothing}}. \end{aligned}$$

Because $\pi > \frac{1}{2}$, the numerators on the LHS of the two inequalities are strictly larger than the numerators on the respective RHS. Therefore, at least one of these inequalities is always satisfied. Without loss, assume it is the first one. Then suppose by way of contradiction that the second inequality is violated. Then

$$\begin{aligned} \frac{\pi \cdot z_{t_{|\mathcal{S}|},G} + z_{t_{|\mathcal{S}|},\varnothing} \cdot r^E_\varnothing(G, t_{|\mathcal{S}|})}{\pi \cdot z_{t_{|\mathcal{S}|},G} + (1 - \pi) \cdot z_{t_{|\mathcal{S}|},B} + z_{t_{|\mathcal{S}|},\varnothing}} &> \frac{(1 - \pi) \cdot z_{t_{|\mathcal{S}|},G} + z_{t_{|\mathcal{S}|},\varnothing} \cdot r^E_\varnothing(G, t_{|\mathcal{S}|})}{\pi \cdot z_{t_{|\mathcal{S}|},B} + (1 - \pi) \cdot z_{t_{|\mathcal{S}|},G} + z_{t_{|\mathcal{S}|},\varnothing}}, \\ \frac{\pi \cdot z_{t_{|\mathcal{S}|},B} + z_{t_{|\mathcal{S}|},\varnothing} \cdot r^E_\varnothing(B, t_{|\mathcal{S}|})}{\pi \cdot z_{t_{|\mathcal{S}|},B} + (1 - \pi) \cdot z_{t_{|\mathcal{S}|},G} + z_{t_{|\mathcal{S}|},\varnothing}} &\leqslant \frac{(1 - \pi) \cdot z_{t_{|\mathcal{S}|},B} + z_{t_{|\mathcal{S}|},\varnothing} \cdot r^E_\varnothing(B, t_{|\mathcal{S}|})}{\pi \cdot z_{t_{|\mathcal{S}|},G} + (1 - \pi) \cdot z_{t_{|\mathcal{S}|},B} + z_{t_{|\mathcal{S}|},\varnothing}}. \end{aligned} \tag{19}$$

Then if we subtract the latter from the former, we get

$$\begin{aligned} &\frac{\pi \cdot z_{t_{|\mathcal{S}|},G} + (1 - \pi) \cdot z_{t_{|\mathcal{S}|},B} + z_{t_{|\mathcal{S}|},\varnothing} \cdot (r^E_\varnothing(G, t_{|\mathcal{S}|}) + r^E_\varnothing(B, t_{|\mathcal{S}|}))}{\pi \cdot z_{t_{|\mathcal{S}|},G} + (1 - \pi) \cdot z_{t_{|\mathcal{S}|},B} + z_{t_{|\mathcal{S}|},\varnothing}} > \\ &\qquad > \frac{\pi \cdot z_{t_{|\mathcal{S}|},B} + (1 - \pi) \cdot z_{t_{|\mathcal{S}|},G} + z_{t_{|\mathcal{S}|},\varnothing} \cdot (r^E_\varnothing(G, t_{|\mathcal{S}|}) + r^E_\varnothing(B, t_{|\mathcal{S}|}))}{\pi \cdot z_{t_{|\mathcal{S}|},B} + (1 - \pi) \cdot z_{t_{|\mathcal{S}|},G} + z_{t_{|\mathcal{S}|},\varnothing}}. \end{aligned}$$

If $r^E_\varnothing(G, t_{|\mathcal{S}|}) + r^E_\varnothing(B, t_{|\mathcal{S}|}) = 1$ then we get a contradiction because then the inequality reduces to $1 > 1$. On the other hand, if $r^E_\varnothing(G, t_{|\mathcal{S}|}) + r^E_\varnothing(B, t_{|\mathcal{S}|}) < 1$, then the inequality above reduces to

$$\frac{1}{\pi \cdot z_{t_{|\mathcal{S}|},G} + (1 - \pi) \cdot z_{t_{|\mathcal{S}|},B} + z_{t_{|\mathcal{S}|},\varnothing}} < \frac{1}{\pi \cdot z_{t_{|\mathcal{S}|},B} + (1 - \pi) \cdot z_{t_{|\mathcal{S}|},G} + z_{t_{|\mathcal{S}|},\varnothing}},$$

which clearly violates the second inequality in (19).

Step 3. If all reports are babbling at $t_{|\mathcal{S}|}$ then $V^E_{t_{|\mathcal{S}|},\eta} = V^Q_{t_{|\mathcal{S}|}}$ for any $\eta \in \{G, B\}$. Indeed, in Case 7 of Step 1 we established that if report m is babbling then $\beta^G(m, t_{|\mathcal{S}|}) = \beta^B(m, t_{|\mathcal{S}|})$. And by (18) we therefore get the result.

Step 4. We next show that if for some $t_i \in \mathcal{S}$, all reports in all periods $t > t_i$ are babbling, then either all reports are babbling in period t_i or $r^E_G(G, t_i) = r^E_B(B, t_i) = 1$. Note that it is enough to verify that incentives of an analyst are the same at t_i as they were at $t_{|\mathcal{S}|}$ in Step 1.

Indeed, by Proposition 1, the quack is indifferent between making either report at t_i and staying silent (and making a report in any further period in $\mathcal{S}$ or not making a report by T if this option is on path). The uninformed expert is indifferent between making either report and staying silent as well. This is because if he makes a report, he gets the same value as the quack because they have the same information. If he stays silent at t_i, his reputation is frozen at β_{t_i} until and after the revelation of the state at T (see Case 7 of Step 1). Therefore, $W^E_{t_i,\varnothing}(\varnothing) = W^Q_{t_i}(\varnothing)$ as well, so the uninformed expert is indeed indifferent between all $m \in \{G, B, \varnothing\}$, same as the quack. Finally, the informed expert with $\eta_{t_i} \in \{G, B\}$ faces a choice between the two reports and staying silent, with the latter, again, freezing his reputation at β_{t_i} for the remainder of the game.

Therefore, by following the same argument as we used for $t_{|\mathcal{S}|}$ in Step 1, we get the result. Note that the statements and the arguments from Steps 2 and 3 then carry over to t_i as well.

Step 5. Using the inductive reasoning we then get that either all reports in all periods within support $\mathcal{S}$ are babbling or there exists $\bar{t}$ such that all reports in all periods $t > \bar{t}$ are babbling and $r^E_G(G, \bar{t}) = r^E_B(B, \bar{t}) = 1$. We next show that $\bar{t}$ is exactly the Godwin point defined by the value equality. First, it follows from Step 3 that $V^E_{\bar{t},\varnothing} = V^Q_{\bar{t}}$. Therefore, it is enough to establish that $V^E_{t,\varnothing} > V^Q_t$ for $t < \bar{t}$. Recall from Step 2 that $V^E_{\bar{t},\eta} > V^Q_{\bar{t}}$ for $\eta \in \{G, B\}$. Take any $t < \bar{t}$ and consider the following strategy for the expert who is uninformed at t: wait until $\bar{t}$ and make report $(\eta_{\bar{t}}, \bar{t})$ if $\eta_{\bar{t}} \in \{G, B\}$, and make a random report if $\eta_{\bar{t}} = \varnothing$. In the former case, the expert gets a strictly higher value than the quack, while in the latter the two are the same. Since the expert faces a strictly positive probability of receiving the private signal between t and $\bar{t}$, this strategy gives the uninformed expert a payoff that is strictly higher than what the quack gets, $V^E_{t,\varnothing} > V^Q_t$, which completes the proof.

Step 6. We next show that $r^E_\varnothing(G, t) = r^E_\varnothing(B, t) = 0$ for $t < \bar{t}$. Suppose the contrary: for some $t < \bar{t}$, the uninformed expert weakly prefers, without loss, report (G, t) over staying silent. But then

$$V^E_{t,\varnothing} = W^E_{t,\varnothing}(G, t) = W^Q_t(G, t) = V^Q_t,$$

because the uninformed expert and the quack have the same information about ω. At the same time, Step 5 above shows that there exists a strategy that grants the t-uninformed expert a payoff strictly higher than the quack's: $V^E_{t,\varnothing} > V^Q_t$, which gives us a contradiction.

Step 7. Finally, we establish that $r^E_G(B, t) = r^E_B(G, t) = 0$ for $t \leqslant \bar{t}$. Suppose the contrary: that for some $t \leqslant \bar{t}$, the expert either prefers to report G when $\eta_t = B$, or to report B when $\eta_t = G$. If this preference is strict, then by (SY) we have $r^E_G(G, t) = r^E_B(B, t) = 0$, which violates (ML). Therefore, the expert should be indifferent between the two reports for both $\eta_t \in \{G, B\}$. It follows from the arguments in Steps 2 and 5 that $V^E_{t,\eta} > V^Q_t$ for any $\eta \in \{G, B\}$ and $t \leqslant \bar{t}$, which means that

$$V^E_{t,G} - V^Q_t = W^E_{t,G}(G, t) - W^Q_t(G, t) > 0,$$
$$V^E_{t,B} - V^Q_t = W^E_{t,B}(G, t) - W^Q_t(G, t) > 0.$$

We have seen in Step 2 that these inequalities are equivalent to

$$\beta^G(G,t) > \beta^B(G,t),$$
$$\beta^B(G,t) > \beta^G(G,t),$$

so the two are clearly at a contradiction. □

Proof of Proposition 3. The proof is valid for all $\pi \in \left(\frac{1}{2},1\right]$. Let $\{r^\gamma_\eta(m,t)\}$ be an equilibrium strategy profile. Consider a new strategy profile $\{\tilde{r}^\gamma_\eta(m,t)\}$ such that $\tilde{r}^E_\eta(m,t) = r^E_\eta(m,t)$, $\tilde{r}^Q(m,t) = r^Q(m,t)$ for all $t \leqslant \bar{t}$ and $\tilde{r}^E_\eta(m,t) = \tilde{r}^Q(m,t) = 0$ for all $t > \bar{t}$. As strategies coincide on $\tilde{\mathcal{S}}$ and all reports (m,t) with $t > \bar{t}$ are babbling in the original equilibrium, the following are true:

1. beliefs $\beta(m,t)$ and $\beta^\omega(m,t)$ induced by the two strategy profiles coincide for all $\omega, m \in \{G,B\}$, $t \in \tilde{\mathcal{S}}$;
2. belief sequences β_t induced by the two strategy profiles coincide for all $t \in \mathcal{T}$.

The latter statement also exploits the fact that $\mathcal{S}\backslash\tilde{\mathcal{S}}$ is nonempty (otherwise the proposition statement trivially holds), so it must be that $r^Q(G,\bar{t}) + r^Q(B,\bar{t}) < 1$ and $r^E(G,\bar{t}) + r^E(B,\bar{t}) < 1$.

The first statement above implies that any report (m,t) with $t \leqslant \bar{t}$ yields the same payoff under either strategy profile. The second statement claims that not making a report in any period yields the same payoffs as well. Strategy of reporting nothing yields the same payoff under $\{\tilde{r}^\gamma_\eta(m,t)\}$ as any report (m,t) with $t > \bar{t}$ under $\{r^\gamma_\eta(m,t)\}$, since all such reports are babbling. Finally, any report (m,t) with $t \notin \mathcal{S}$ yields the same payoff under either strategy profile due to (OP).

Everything said above directly implies that for all γ and η, if $r^\gamma_\eta(m,t)$ is a best response for type-γ analyst to strategy profile $\{r^\gamma_\eta(m,t)\}$ then $\tilde{r}^\gamma_\eta(m,t)$ is a best response to strategy profile $\{\tilde{r}^\gamma_\eta(m,t)\}$ and yields the same payoff. □

Corollary 12. *In any informative equilibrium:*

1. $z_{t,G} = z_{t,B}$ *for all* $t \in \mathcal{S}$,
2. $\rho_t = \rho_0$ *for all* $t \in \mathcal{S}$,
3. $\beta_{\bar{t}} = \beta^\omega(\varnothing)$, *i.e., if no report was made, then the state revelation does not affect the analyst's reputation.*

Proof. First, (SY) states that $r^E_G(G,t) = r^E_B(B,t)$. From Proposition 2 we additionally infer that $r^E_G(B,t) = r^E_B(G,t) = 0$ for all $t \in \mathcal{S}$ because the equilibrium is informative. Therefore, for all $t \in \mathcal{S}$ we have

$$\sum_m r^E_G(m,t) = \sum_m r^E_B(m,t).$$

The same is true for all $t \notin \mathcal{S}$, since $r^E_G(m,t) = r^E_B(m,t) = 0$ for both $m \in \{G,B\}$ in those periods. From (14) we then get that $z_{t,G} = z_{t,B}$ for all $t \in \mathcal{T}$. This proves the first part of the claim.

Next, we can write the belief update about the state in case no report was made:

$$\rho_{t+1} = \rho_t \cdot \frac{1-(1-b_{t-1})\cdot\left(r^Q(G,t)+r^Q(B,t)\right) - b_{t-1}\cdot\mathbb{E}_\eta\left[r^E_\eta(G,t)+r^E_\eta(B,t)|\omega=G\right]}{1-(1-b_{t-1})\cdot\left(r^Q(G,t)+r^Q(B,t)\right) - b_{t-1}\cdot\mathbb{E}_\eta\left[r^E_\eta(G,t)+r^E_\eta(B,t)|\omega=B\right]}. \tag{20}$$

Again, remember that $r^E_G(B,t) = r^E_B(G,t) = 0$ and $z_{t,G} = z_{t,B}$. These two equalities and (SY) automatically show us that the numerator and the denominator are equal in (20) due to (13). This proves the second part of the claim.

Finally, we establish that $\beta_{\bar{t}} = \beta^\omega(\varnothing)$. Given (4) and (5), we need to verify that for any $\omega \in \{G, B\}$,

$$\frac{1 - r^E(G,\bar{t}) - r^E(B,\bar{t})}{1 - r^Q(G,\bar{t}) - r^Q(B,\bar{t})} = \frac{1 - \mathbb{E}_\eta[r^E_\eta(G,\bar{t}) + r^E_\eta(B,\bar{t})|\omega]}{1 - r^Q(G,\bar{t}) - r^Q(B,\bar{t})}. \tag{21}$$

Given (13), (SY), and $r^E_G(B,\bar{t}) = r^E_B(G,\bar{t}) = 0$, we get

$$\mathbb{E}_\eta[r^E_\eta(G,\bar{t}) + r^E_\eta(B,\bar{t})|\omega] = \frac{z_{\bar{t}} \cdot r^E_\omega(\omega,\bar{t}) + z_{\bar{t},\varnothing} \cdot r^E_\varnothing(G,\bar{t}) + z_{\bar{t},\varnothing} \cdot r^E_\varnothing(B,\bar{t})}{z_{\bar{t}} + z_{\bar{t},\varnothing}} = r^E(G,\bar{t}) + r^E(B,\bar{t}),$$

where $z_{\bar{t}} := z_{\bar{t},G} = z_{\bar{t},B}$. The denominators in (21) are also the same, which completes the proof. □

Proof of the Main Result

This section presents the proof of the main result, Theorem 1. To avoid duplicating the arguments, we merge it together with the proofs of Proposition 6 and the relevant part of Proposition 8. Before we proceed to the proof proper, we provide some expressions for belief updating that will be useful in the proof.

Using Proposition 2, we can rewrite expressions (4) and (6) in a more explicit form. Proposition 2 implies that for all $t \leqslant \bar{t}$ we have $r^E_G(B,t) = r^E_B(G,t) = 0$ and $r^E_\varnothing(G,t) = r^E_\varnothing(B,t) = 0$ for all $t < \bar{t}$. Moreover, due to Corollary 12, we can introduce the notion of $z_t := z_{t,G} = z_{t,B}$, and, therefore, we can define

$$z^*_t := \frac{z_t}{z_t + z_\varnothing}. \tag{22}$$

Then (13) implies that for all $t \in \mathcal{S}$, we have

$$\begin{aligned} \beta(G,t) &= \beta_{t-1} \cdot \frac{r^E(G,t)}{r^Q(G,t)} = \beta_{t-1} \cdot \frac{p^*_0 \cdot z^*_t \cdot r^E_G(G,t) + (1 - z^*_t) \cdot r^E_\varnothing(G,t)}{r^Q(G,t)}, \\ \beta(B,t) &= \beta_{t-1} \cdot \frac{r^E(B,t)}{r^Q(B,t)} = \beta_{t-1} \cdot \frac{(1 - p^*_0) \cdot z^*_t \cdot r^E_B(B,t) + (1 - z^*_t) \cdot r^E_\varnothing(B,t)}{r^Q(B,t)}, \end{aligned} \tag{23}$$

as well as

$$\begin{aligned} \beta^G(G,t) &= \beta_{t-1} \cdot \frac{\mathbb{E}_\eta[r^E_\eta(G,t)|G]}{r^Q(G,t)} = \beta_{t-1} \cdot \frac{\pi \cdot z^*_t \cdot r^E_G(G,t) + (1 - z^*_t) \cdot r^E_\varnothing(G,t)}{r^Q(G,t)}, \\ \beta^B(B,t) &= \beta_{t-1} \cdot \frac{\mathbb{E}_\eta[r^E_\eta(B,t)|B]}{r^Q(B,t)} = \beta_{t-1} \cdot \frac{\pi \cdot z^*_t \cdot r^E_B(B,t) + (1 - z^*_t) \cdot r^E_\varnothing(B,t)}{r^Q(B,t)}, \\ \beta^B(G,t) &= \beta_{t-1} \cdot \frac{\mathbb{E}_\eta[r^E_\eta(G,t)|B]}{r^Q(G,t)} = \beta_{t-1} \cdot \frac{(1-\pi) \cdot z^*_t \cdot r^E_G(G,t) + (1 - z^*_t) \cdot r^E_\varnothing(G,t)}{r^Q(G,t)}, \\ \beta^G(B,t) &= \beta_{t-1} \cdot \frac{\mathbb{E}_\eta[r^E_\eta(B,t)|G]}{r^Q(B,t)} = \beta_{t-1} \cdot \frac{(1-\pi) \cdot z^*_t \cdot r^E_B(B,t) + (1 - z^*_t) \cdot r^E_\varnothing(B,t)}{r^Q(B,t)}. \end{aligned} \tag{24}$$

It is also worth remembering that $r^E_\varnothing(m,t) = 0$ for any $t < \bar{t}$ (see Proposition 2).

In case no report was made in period $t < \bar{t}$, the belief is updated as

$$\beta_t = \beta_{t-1} \cdot \frac{1 - r^E(G,t) - r^E(B,t)}{1 - r^Q(G,t) - r^Q(B,t)} = \beta_{t-1} \cdot \frac{1 - z^*_t \cdot r^E_G(G,t)}{1 - r^Q(G,t) - r^Q(B,t)}, \tag{25}$$

where we use that $r^E_G(G,\bar{t}) = r^E_B(B,\bar{t}) = 1$ by Proposition 2. The analogous expression for $t = \bar{t}$ is given by

$$\beta_{\bar{t}} = \beta_{t_{|\mathcal{S}|-1}} \cdot \frac{1 - r^E(G,\bar{t}) - r^E(B,\bar{t})}{1 - r^Q(G,\bar{t}) - r^Q(B,\bar{t})} = \beta_{t_{|\mathcal{S}|-1}} \cdot \frac{\left(1 - z^*_{\bar{t}}\right) \cdot \left(1 - r^E_\varnothing(G,\bar{t}) - r^E_\varnothing(B,\bar{t})\right)}{1 - r^Q(G,\bar{t}) - r^Q(B,\bar{t})}, \tag{26}$$

and by Corollary 12, we have that $\beta^{\omega}(\varnothing) = \beta_{\bar{t}}$ for both $\omega \in \{G, B\}$.

Finally, due to (15), the informativeness measure can be rewritten as

$$i(m,t) = \ln\left(1+\beta^{G}(m,t)\right) - \ln\left(1+\beta^{B}(m,t)\right).$$

We now proceed to the proof.

Proof of Theorem 1 and Proposition 6 (applies also under the conditions stated in Proposition 8). First, recall that Propositions 1, 2 and 3 are valid for all $\pi \in \left(\frac{1}{2}, 1\right]$, and so can be employed in this proof. Further, note that in all babbling periods t we have $i(m,t) = 0$, $\beta(m,t) = \beta_{t-1}$ for $m \in \{G, B\}$, and β_t stays on a constant level. Together with Propositions 2 and 3 this means that it is enough to show the statement of the Theorem for informative equilibria, where we have $\mathcal{S} = \{t_1, t_2, \ldots, t_{|\mathcal{S}|-1}, t_{|\mathcal{S}|} = \bar{t}\}$. The proof is separated into several steps.

Step 1. We start by introducing $\Delta w_{\eta}(m,t)$, the premium for guessing the state correctly, defined for all $m, \eta \in \{G, B\}$ and $t \in \mathcal{T}$ as

$$\Delta w_{\eta}(m,t) := w^{c}\left(\beta^{\eta}(m,t)\right) - w^{c}\left(\beta^{-\eta}(m,t)\right),$$

and showing that it is a weakly decreasing function of t on $\mathcal{S}$ given $m = \eta$ (note that $\Delta w_G(m,t) = -\Delta w_B(m,t)$). Fix some arbitrary $t \in \mathcal{S}$. Suppose the expert has private information $\eta_t = G$ at time t, but has not yet made any report. He chooses a report (m, τ) with $\tau \geqslant t$ to maximize $W^{E}_{t,G}(m,\tau)$.[32] Expanding $W^{E}_{t,G}(m,\tau)$, we get the following expression:

$$\sum_{s=t}^{\tau-1} w\left(\beta_s\right) + \sum_{s=\tau}^{T-1} w\left(\beta(m,\tau)\right) + \frac{\pi p_0}{p_0^*} \cdot w^{c}\left(\beta^{G}(m,\tau)\right) + \left(1 - \frac{\pi p_0}{p_0^*}\right) \cdot w^{c}\left(\beta^{B}(m,\tau)\right),$$

where $p_0^* = p_0 \cdot \pi + (1-p_0) \cdot (1-\pi)$.

A quack is, in equilibrium, indifferent between all such reports at time t. This means that his continuation value $W^{Q}_{t}(m,\tau)$, which can similarly be written as

$$\sum_{s=t}^{\tau-1} w\left(\beta_s\right) + \sum_{s=\tau}^{T-1} w\left(\beta(m,\tau)\right) + p_0 \cdot w^{c}\left(\beta^{G}(m,\tau)\right) + (1-p_0) \cdot w^{c}\left(\beta^{B}(m,\tau)\right),$$

is constant over all (m,τ) for a given t. Therefore, the optimization problem of the expert with $\eta = G$ becomes equivalent to maximizing the difference $\Delta w_G(m,\tau)$ over all $\tau \in \{\mathcal{S} | \tau \geqslant t\}$ and $m \in \{G, B\}$. Similarly, the expert with $\eta_t = B$ chooses report (m,τ) which maximizes $\Delta w_B(m,\tau)$. Propositions 1, 2 and 3 imply that $\mathcal{S} = \{t \in \mathcal{T} \,|\, r^{E}_{\eta}(\eta,t) > 0\}$ for any $\eta \in \{G, B\}$, so since $t \in \mathcal{S}$, it must be that (G,t) maximizes $\Delta w_G(m,\tau)$, and (B,t) maximizes $\Delta w_B(m,\tau)$ across all (m,τ) with $\tau \in \{\mathcal{S} \,|\, \tau \geqslant t\}$. Because t was chosen arbitrary, it implies that $\Delta w_{\eta}(\eta,t)$ should be a weakly decreasing function of t on $\mathcal{S}$.

Step 2. The second step of the proof shows that $\Delta w_{\eta}(\eta,t)$ is (strictly) decreasing on $\mathcal{S}\backslash\{\bar{t}\}$ if and only if $\beta^{\eta}(\eta,t)$ is (strictly) decreasing on $\mathcal{S}\backslash\{\bar{t}\}$ (and if an equilibrium is reticent, then $\Delta w_{\eta}(\eta,t)$ is (strictly) decreasing on $\mathcal{S}$ if and only if $\beta^{\eta}(\eta,t)$ is (strictly) decreasing on $\mathcal{S}$). We demonstrate this for all cases stated in Theorem 1, Proposition 6, and Proposition 8 separately.

Case 1: $\pi = 1$.

[32] Remember that by Proposition 2, if the expert has private signal, then he makes a report by period $\bar{t}$.

This case is obvious, as then $\beta^{-\eta}(\eta,t) = w^c(\beta^{-\eta}(\eta,t)) = 0$ for any $\eta \in \{G,B\}$, so $\Delta w_\eta(\eta,t) = w^c(\beta^\eta(\eta,t))$, and $w^c(\cdot)$ is a strictly increasing function.

Case 2: $\pi < 1$ and $w^c(\cdot)$ is convex.

Note that since $r^E_\varnothing(m,t) = 0$ for all $t \in \mathcal{S}\backslash\{\bar{t}\}$ (or for all $t \in \mathcal{S}$ if an equilibrium is reticent), from (24) we have

$$\beta^{-\eta}(\eta,t) = \frac{1-\pi}{\pi}\cdot\beta^\eta(\eta,t) \qquad \Rightarrow \qquad \Delta w_\eta(\eta,t) = w^c(\beta^\eta(\eta,t)) - w^c\left(\frac{1-\pi}{\pi}\cdot\beta^\eta(\eta,t)\right), \tag{27}$$

where $\frac{1-\pi}{\pi} \in (0,1)$ because $\pi > \frac{1}{2}$. Take any $\tau_1 < \tau_2$ with $\tau_1,\tau_2 \in \mathcal{S}$ and let $x_1 := \beta^\eta(\eta,\tau_1)$, $x_2 := \beta^\eta(\eta,\tau_2)$. Because $w^c(x)$ is convex,

$$w^c(x) - w^c\left(\frac{1-\pi}{\pi}\cdot x\right)$$

is an increasing function of x. Therefore, the inequality

$$w^c(x_1) - w^c\left(\frac{1-\pi}{\pi}\cdot x_1\right) \geqslant w^c(x_2) - w^c\left(\frac{1-\pi}{\pi}\cdot x_2\right)$$

is equivalent to $x_1 \geqslant x_2$, i.e., $\beta^\eta(\eta,\tau_1) \geqslant \beta^\eta(\eta,\tau_2)$.

Case 3: $\pi \in \left(\frac{\bar{d}}{\underline{d}+\bar{d}}, 1\right)$ and $\frac{dw^c(\beta)}{d\beta} \in [\underline{d},\bar{d}]$.

Note that (27) still applies. Similarly to the previous case, take any $\tau_1 < \tau_2$ with $\tau_1,\tau_2 \in \mathcal{S}$ and let $x_1 := \beta^\eta(\eta,\tau_1)$, $x_2 := \beta^\eta(\eta,\tau_2)$. Suppose $w^c(x_1) - w^c\left(\frac{1-\pi}{\pi}\cdot x_1\right) \geqslant w^c(x_2) - w^c\left(\frac{1-\pi}{\pi}\cdot x_2\right)$. Then we show that it must be that $x_1 \geqslant x_2$. Assume the converse – that $x_1 < x_2$. Then

$$\begin{aligned} 0 &\leqslant \left(w^c(x_1) - w^c\left(\frac{1-\pi}{\pi}\cdot x_1\right)\right) - \left(w^c(x_2) - w^c\left(\frac{1-\pi}{\pi}\cdot x_2\right)\right) \leqslant \\ &\leqslant -\underline{d}\cdot(x_2-x_1) + \bar{d}\cdot\left(\frac{1-\pi}{\pi}\cdot x_2 - \frac{1-\pi}{\pi}\cdot x_1\right) = (x_2-x_1)\cdot\left(\bar{d}\cdot\frac{1-\pi}{\pi} - \underline{d}\right), \end{aligned} \tag{28}$$

where the latter inequality follows from the bounds on the derivative of w^c. The assumption $\pi > \frac{\bar{d}}{\underline{d}+\bar{d}}$ implies $\bar{d}\cdot\frac{1-\pi}{\pi} - \underline{d} < 0$, which directly contradicts (28). We conclude that if $w^c(x_1) - w^c\left(\frac{1-\pi}{\pi}\cdot x_1\right) \geqslant w^c(x_2) - w^c\left(\frac{1-\pi}{\pi}\cdot x_2\right)$, then $x_1 \geqslant x_2$.

Conversely, if $x_1 \geqslant x_2$, then

$$0 \leqslant (x_1-x_2)\cdot\left(\underline{d} - \bar{d}\cdot\frac{1-\pi}{\pi}\right) \leqslant \left(w^c(x_1) - w^c\left(\frac{1-\pi}{\pi}\cdot x_1\right)\right) - \left(w^c(x_2) - w^c\left(\frac{1-\pi}{\pi}\cdot x_2\right)\right),$$

which gives the result.

Step 3. We next show that whenever $|\mathcal{S}| \geqslant 3$ and an equilibrium on $\mathcal{S}$ exists, it must be that $\beta_{t_1} \geqslant \beta(m,t_1)$ for any $m \in \{G,B\}$ (alternatively, if the equilibrium is reticent, then this claim is valid for any $\mathcal{S}$ with $|\mathcal{S}| \geqslant 2$). Assume the contrary, i.e., there exists $m \in \{G,B\}$ such that $\beta_{t_1} < \beta(m,t_1)$. The quack's values from reports (m,t_1) and (m,t_2) evaluated at t_1 are equal to

$$\begin{aligned} W^Q_{t_1}(m,t_1) &= (T-t_1)\cdot w(\beta(m,t_1)) + p_0\cdot w^c\left(\beta^G(m,t_1)\right) + (1-p_0)\cdot w^c\left(\beta^B(m,t_1)\right), \\ W^Q_{t_1}(m,t_2) &= (t_2-t_1)\cdot w(\beta_{t_1}) + (T-t_2)\cdot w(\beta(m,t_2)) + p_0\cdot w^c\left(\beta^G(m,t_2)\right) + (1-p_0)\cdot w^c\left(\beta^B(m,t_2)\right). \end{aligned}$$

As $w(\cdot)$ is strictly increasing, and $\beta_{t_1} < \beta(m,t_1)$, $W^Q_{t_1}(m,t_1) = W^Q_{t_1}(m,t_2)$ implies

$$(T-t_2)\cdot w\left(\beta(m,t_1)\right) + p_0\cdot w^c\left(\beta^G(m,t_1)\right) + (1-p_0)\cdot w^c\left(\beta^B(m,t_1)\right) < \\ < (T-t_2)\cdot w\left(\beta(m,t_2)\right) + p_0\cdot w^c\left(\beta^G(m,t_2)\right) + (1-p_0)\cdot w^c\left(\beta^B(m,t_2)\right).$$

Consequently, it must be that either $\beta(m,t_1) < \beta(m,t_2)$, or $\beta^m(m,t_1) < \beta^m(m,t_2)$, or $\beta^{-m}(m,t_1) < \beta^{-m}(m,t_2)$. However, (23) and (24) imply that both $\beta^G(m,t)$ and $\beta^B(m,t)$ differ from $\beta(m,t)$ by a constant factor for any $t \in \mathcal{S}\backslash\{\bar{t}\}$ since $r^E_{\varnothing}(m,t) = 0$, so the three inequalities are equivalent. Therefore, it must be that $\beta^m(m,t_1) < \beta^m(m,t_2)$, which contradicts $\beta^m(m,t)$ being weakly decreasing on $\mathcal{S}\backslash\{\bar{t}\}$. In reticent equilibria $r^E_{\varnothing}(m,\bar{t}) = 0$, therefore, the claim extends to $\bar{t}$ as well.

Step 4. We finally show how the claim in the theorem can be obtained from the previous steps. We have shown that $\beta_{t_1} \geqslant \beta(m,t_1)$ for any $m \in \{G,B\}$. Consequently, by the martingale property of beliefs, we have that $\beta_{t_1} \geqslant \beta_0$ and $\beta(m,t_1) \leqslant \beta_0$ for at least one $m \in \{G,B\}$. For both $m \in \{G,B\}$: as $\beta(m,t_1) \leqslant \beta_{t_1}$, to make the quack indifferent between reports (m,t_1) and (m,t_2) we must have that either $\beta(m,t_2) \leqslant \beta(m,t_1)$, or $\beta^m(m,t_2) \leqslant \beta^m(m,t_1)$, or $\beta^{-m}(m,t_2) \leqslant \beta^{-m}(m,t_1)$. Again, (23) and (24) imply that all three inequalities are equivalent, so all three have to hold simultaneously. The martingale property of beliefs together with the resulting inequalities $\beta_{t_1} \geqslant \beta(m,t_1) \geqslant \beta(m,t_2)$ for $m \in \{G,B\}$ imply $\beta_{t_2} \geqslant \beta_{t_1}$. By iterating this argument, we conclude that $\beta(m,t) \leqslant \beta_t$ and β_t is increasing in t on $\mathcal{S}\backslash\{\bar{t}\}$ (on the whole $\mathcal{S}$ if the equilibrium is reticent).

The above proves the second and the third parts of Theorem 1 and Proposition 6.[33] It remains to show the first part. Note that, by the same inductive reasoning as above, if $\beta_{t_1} \geqslant \beta(m,t_1)$ then $\beta_{t_{|\mathcal{S}|-1}} \geqslant \beta(m,t_{|\mathcal{S}|-1})$. Consequently, it is possible to show that $\beta^m(m,\bar{t}) \leqslant \beta^m(m,t_{|\mathcal{S}|-1})$. Indeed, suppose the converse: $\beta^m(m,\bar{t}) > \beta^m(m,t_{|\mathcal{S}|-1})$. Then to make the quack indifferent between reporting m at $t_{|\mathcal{S}|-1}$ and at $\bar{t}$, we must have $\beta^{-m}(m,\bar{t}) < \beta^{-m}(m,t_{|\mathcal{S}|-1})$. But then

$$w^c\left(\beta^m(m,t_{|\mathcal{S}|-1})\right) - w^c\left(\beta^{-m}(m,t_{|\mathcal{S}|-1})\right) < w^c\left(\beta^m(m,\bar{t})\right) - w^c\left(\beta^{-m}(m,\bar{t})\right),$$

which contradicts the fact that $\Delta w_\eta(m,\tau)$ is weakly decreasing in t on $\mathcal{S}$ for $m = \eta$.

Finally, recall that for all $t \in \mathcal{S}\backslash\{\bar{t}\}$ we have

$$|i(m,t)| = \ln\left(\frac{1+\beta^m(m,t)}{1+\beta^{-m}(m,t)}\right) = \ln\left(\frac{1+\beta^m(m,t)}{1+\frac{1-\pi}{\pi}\cdot\beta^m(m,t)}\right), \tag{29}$$

which is then a decreasing function of t on $\mathcal{S}\backslash\{\bar{t}\}$ because $\ln(1+x) - \ln\left(1+\frac{1-\pi}{\pi}x\right)$ is an increasing function of x for $\pi \in \left(\frac{1}{2},1\right]$. For the last two points of $\mathcal{S}$ we have

$$|i(m,t_{|\mathcal{S}|-1})| - |i(m,\bar{t})| = \ln\left(\frac{1+\beta^m(m,t_{|\mathcal{S}|-1})}{1+\frac{1-\pi}{\pi}\cdot\beta^m(m,t_{|\mathcal{S}|-1})}\right) - \ln\left(\frac{1+\beta^m(m,\bar{t})}{1+\frac{1-\pi}{\pi}\cdot\beta^m(m,\bar{t}) + \frac{2\pi-1}{\pi}\cdot\frac{(1-z^*_{\bar{t}})\cdot r^E_{\varnothing}(m,\bar{t})}{r^Q(m,\bar{t})}}\right) \geqslant 0,$$

where the last inequality follows from $\beta^m(m,t_{|\mathcal{S}|-1}) \geqslant \beta^m(m,\bar{t})$ and the fact that $\pi > \frac{1}{2}$. This concludes the proof of Theorem 1 and Proposition 6. □

[33]The statement that β_t is constant on $\mathcal{T}\backslash\mathcal{S}$ follows trivially from (4).

Proofs for Sections 5.5 and 5.6

Proving statements about equilibrium existence and properties requires introducing additional notation and showing some supplementary results first. In particular, consider the following classification:

Definition. *We call an informative equilibrium:*

1. *a* relay equilibrium *if* $r^E_\eta(\eta, t) = 1$ *for* $\eta \in \{G, B\}$ *and for all* $t \in \mathcal{S}$*;*
2. *a* delay equilibrium *otherwise.*

We proceed with describing which conditions are necessary for a given profile of strategies $\{r^\gamma_\eta(m,t)\}$ to constitute a *relay* equilibrium. We consider two sub-cases depending on whether staying silent at $\bar{t}$ is on the equilibrium path or not. Also because we consider informative equilibria, in all further statements we have $t_{|\mathcal{S}|} = \bar{t}$.

Lemma 13. *1. Strategy profile* $\{r^\gamma_\eta(m,t)\}$ *with given* $r^E_\varnothing(G,\bar{t}) + r^E_\varnothing(B,\bar{t}) < 1$ *constitutes a relay equilibrium on* $\mathcal{S}$ *only if*

$$W^Q_{t_1}(m,t) = W^Q_{t_1}(\varnothing) = \bar{W} \text{ for all } t \in \mathcal{S} \text{ and } m \in \{G, B\} \text{ for some } \bar{W} \in \mathbb{R}_+. \tag{30}$$

Moreover, there exists at most one profile $\{r^\gamma_\eta(m,t)\}$ *which constitutes a relay equilibrium and solves system* (30)*.*

2. Strategy profile $\{r^\gamma_\eta(m,t)\}$ *with given* $r^E_\varnothing(G,\bar{t}) + r^E_\varnothing(B,\bar{t}) = 1$ *constitutes a relay equilibrium on* $\mathcal{S}$ *only if*

$$\begin{aligned} &W^Q_{t_1}(m,t) = \bar{W} \text{ for all } t \in \mathcal{S} \text{ and } m \in \{G, B\} \text{ for some } \bar{W} \in \mathbb{R}_+, \\ &r^Q(G,\bar{t}) + r^Q(B,\bar{t}) = 1. \end{aligned} \tag{31}$$

Moreover, there exists at most one profile $\{r^\gamma_\eta(m,t)\}$ *which constitutes a relay equilibrium and solves system* (31)*.*

Proof. By Proposition 1, a strategy profile constitutes an equilibrium only if $W^Q_{t_1}(m,t)$ is constant for all $t \in \mathcal{S}$ and $m \in \{G, B\}$. Additionally, if $r^E_\varnothing(G,\bar{t}) + r^E_\varnothing(B,\bar{t}) < 1$ – that is, not making a report by period $\bar{t}$ is an on-path action – the value that the quack receives from making any report must be equal to the value from making no report.

The proof of Proposition 1 argued that $r^E(G,\bar{t}) + r^E(B,\bar{t}) = 1$ implies $r^Q(G,\bar{t}) + r^Q(B,\bar{t}) = 1$. From Proposition 2 we know that $r^E_G(G,\bar{t}) = r^E_B(B,\bar{t}) = 1$, and, therefore, $r^E(G,\bar{t}) + r^E(B,\bar{t}) = 1$ is equivalent to $r^E_\varnothing(G,\bar{t}) + r^E_\varnothing(B,\bar{t}) = 1$. This completes the proof of the first parts of both statements.

To prove the uniqueness of solutions to the respective systems, assume that there exist two different relay equilibria $\left(\{r^\gamma_\eta(m,t)\}, \beta, \rho\right)$ and $\left(\{\tilde{r}^\gamma_\eta(m,t)\}, \tilde{\beta}, \tilde{\rho}\right)$ such that the respective value functions for the quack, $W^Q_t(m,\tau)$ and $\tilde{W}^Q_t(m,\tau)$, both solve either system (30) or system (31). Note first that the expert's strategy must coincide across the two equilibria, since:

1. $r^E_\varnothing(m,t) = \tilde{r}^E_\varnothing(m,t) = 0$ for all $t \in \mathcal{S}\backslash\{\bar{t}\}$ by Proposition 2;
2. $r^E_\varnothing(G,\bar{t})$ and $r^E_\varnothing(B,\bar{t})$ are fixed by the statement of the lemma;
3. $r^E_\eta(m,t) = \tilde{r}^E_\eta(m,t)$ for $\eta \in \{G, B\}$ and all $t \in \mathcal{S}$ by the assumption that we are dealing with relay equilibria.

Therefore, the difference between the equilibria must come from the quack's strategy. We now show that it has to be that $r^Q(G,t_1) \neq \tilde{r}^Q(G,t_1)$, as otherwise the equilibria coincide. In particular, if $r^Q(G,t_1) = \tilde{r}^Q(G,t_1)$, then (23) and (24) imply that $\beta(G,t_1) = \tilde{\beta}(G,t_1)$ and $\beta^\omega(G,t_1) = \tilde{\beta}^\omega(G,t_1)$, meaning that $W^Q_{t_1}(G,t_1) = \tilde{W}^Q_{t_1}(G,t_1)$. By the first part of the lemma for either case, values $W^Q_{t_1}(m,t)$ and $\tilde{W}^Q_{t_1}(m,t)$ should then coincide for all m and $t \in \mathcal{S}$. Equality $W^Q_{t_1}(B,t_1) = \tilde{W}^Q_{t_1}(B,t_1)$ then implies that $r^Q(B,t_1) = \tilde{r}^Q(B,t_1)$. The latter also yields $\beta_{t_1} = \tilde{\beta}_{t_1}$. By induction on t, we then conclude from $W^Q_{t_1}(m,t) = \tilde{W}^Q_{t_1}(m,t)$ that $r^Q(m,t) = \tilde{r}^Q(m,t)$ for all (m,t) – a contradiction to the assumption that the equilibria are distinct.

Without loss, assume $r^Q(G,t_1) > \tilde{r}^Q(G,t_1)$. Then $\beta(G,t_1) < \tilde{\beta}(G,t_1)$ and $\beta^\omega(G,t_1) < \tilde{\beta}^\omega(G,t_1)$. Because $W^Q_{t_1}(G,t_1) = W^Q_{t_1}(B,t_1)$ and $\tilde{W}^Q_{t_1}(G,t_1) = \tilde{W}^Q_{t_1}(B,t_1)$ we must have $r^Q(B,t_1) > \tilde{r}^Q(B,t_1)$ as well. By (25), this implies that $\beta_{t_1} > \tilde{\beta}_{t_1}$. Consequently, to maintain $W^Q_{t_1}(m,t_1) = W^Q_{t_1}(m,t_2)$ and $\tilde{W}^Q_{t_1}(m,t_1) = \tilde{W}^Q_{t_1}(m,t_2)$ for $m \in \{G,B\}$, it must be that $r^Q(G,t_2) > \tilde{r}^Q(G,t_2)$ and $r^Q(B,t_2) > \tilde{r}^Q(B,t_2)$. Iterating this argument further, we obtain that

$$r^Q(G,\bar{t}) + r^Q(B,\bar{t}) > \tilde{r}^Q(G,\bar{t}) + \tilde{r}^Q(B,\bar{t}). \tag{32}$$

In the context of the first part of the lemma (case $r^E_\varnothing(G,\bar{t}) + r^E_\varnothing(B,\bar{t}) < 1$), (32) implies $\beta_{\bar{t}} > \tilde{\beta}_{\bar{t}}$ and, therefore, $W^Q_{t_1}(\varnothing) > \tilde{W}^Q_{t_1}(\varnothing)$, because the payoff that the quack receives from staying silent is point-wise lower in the former equilibrium. At the same time, because $r^Q(G,t_1) > \tilde{r}^Q(G,t_1)$, we must have $W^Q_{t_1}(G,t_1) < \tilde{W}^Q_{t_1}(G,t_1)$. Since it must be that $W^Q_{t_1}(\varnothing) = W^Q_{t_1}(G,t_1)$ and $\tilde{W}^Q_{t_1}(\varnothing) = \tilde{W}^Q_{t_1}(G,t_1)$, we arrive at a contradiction. In the context of the second part of the lemma (case $r^E_\varnothing(G,\bar{t}) + r^E_\varnothing(B,\bar{t}) = 1$), (32) clearly violates $r^Q(G,\bar{t}) + r^Q(B,\bar{t}) = \tilde{r}^Q(G,\bar{t}) + \tilde{r}^Q(B,\bar{t}) = 1$.

Therefore, $r^Q(G,t_1) > \tilde{r}^Q(G,t_1)$ is not possible in either of the two cases presented in the statement of the lemma. Therefore, given $\mathcal{S}$ and $r^E_\varnothing(G,\bar{t}), r^E_\varnothing(B,\bar{t})$, if a relay equilibrium exists, it is unique. □

The bottom line of the lemma above is that for a given tuple $\left[\mathcal{S}, r^E_\varnothing(G,\bar{t}), r^E_\varnothing(B,\bar{t})\right]$, a strategy profile that constitutes a relay equilibrium is a unique solution to a particular system of algebraic equations. Representing a strategy profile as a solution to a system of equations allows us to compare equilibrium strategies and, therefore, report informativeness across different relay equilibria employing the arguments similar to the Implicit Function Theorem.

In all further lemmas, it is assumed that the strategy profile $r^\gamma_\eta(m,t)$ and all associated equilibrium objects, such as value function $W^Q_t(m,\tau)$, belief profile β, and informativeness measure $i(m,t)$ constitute a solution to either system (30), or system (31) for a given tuple $\left[\mathcal{S}, r^E_\varnothing(G,\bar{t}), r^E_\varnothing(B,\bar{t})\right]$, and therefore are understood as functions of $\left[\mathcal{S}, r^E_\varnothing(G,\bar{t}), r^E_\varnothing(B,\bar{t})\right]$.

The two following lemmas consider the special case when $w(\beta) = \beta^\alpha$ and $w^c(\beta) = \theta \cdot \beta^\alpha$ for some $\alpha, \theta > 0$. The following lemma establishes that in this special case, a relay equilibrium always exists.

Lemma 14. *Suppose $w(\beta) = \beta^\alpha$ and $w^c(\beta) = \theta \cdot \beta^\alpha$ for some $\alpha, \theta > 0$. Fix some $\mathcal{S} \subseteq \mathcal{T}$ and $r^E_\varnothing(G,\bar{t}), r^E_\varnothing(B,\bar{t}) \in [0,1]$. Then:*

1. *if $r^E_\varnothing(G,\bar{t}) + r^E_\varnothing(B,\bar{t}) < 1$, a solution $\{r^\gamma_\eta(m,t)\}$ to system* (30) *always exists;*
2. *if $r^E_\varnothing(G,\bar{t}) + r^E_\varnothing(B,\bar{t}) = 1$, a solution $\{r^\gamma_\eta(m,t)\}$ to system* (31) *always exists.*

Proof. Consider first the case $r^E_\varnothing(G,\bar{t}) + r^E_\varnothing(B,\bar{t}) = 1$. The second part of Lemma 13 implies that $W^Q_{\bar{t}}(G,\bar{t}) = W^Q_{\bar{t}}(B,\bar{t})$. We can then explicitly solve this system defined by these two equations for $r^Q(G,\bar{t})$

and $r^Q(B,\bar{t})$ by invoking (12), (23), and (24). The resulting solution is given by

$$\begin{aligned} r^Q(G,\bar{t}) &= \frac{M_G\left(r^E_{\varnothing}(G,\bar{t})\right)^{\frac{1}{\alpha}}}{M_G\left(r^E_{\varnothing}(G,\bar{t})\right)^{\frac{1}{\alpha}} + M_B\left(r^E_{\varnothing}(B,\bar{t})\right)^{\frac{1}{\alpha}}}, \\ r^Q(B,\bar{t}) &= \frac{M_B\left(r^E_{\varnothing}(B,\bar{t})\right)^{\frac{1}{\alpha}}}{M_G\left(r^E_{\varnothing}(G,\bar{t})\right)^{\frac{1}{\alpha}} + M_B\left(r^E_{\varnothing}(B,\bar{t})\right)^{\frac{1}{\alpha}}}, \end{aligned} \tag{33}$$

where

$$M_G(x) := (T-\bar{t})\cdot\left(p_0^* z_{\bar{t}}^* + (1-z_{\bar{t}}^*)\,x\right)^{\alpha} + \theta\cdot p_0\cdot\left(\pi z_{\bar{t}}^* + (1-z_{\bar{t}}^*)\,x\right)^{\alpha} + \theta\cdot(1-p_0)\cdot\left((1-\pi)\,z_{\bar{t}}^* + (1-z_{\bar{t}}^*)\,x\right)^{\alpha},$$
$$M_B(x) := (T-\bar{t})\cdot\left((1-p_0^*)\,z_{\bar{t}}^* + (1-z_{\bar{t}}^*)\,x\right)^{\alpha} + \theta\cdot p_0\cdot\left((1-\pi)\,z_{\bar{t}}^* + (1-z_{\bar{t}}^*)\,x\right)^{\alpha} + \theta\cdot(1-p_0)\cdot\left(\pi z_{\bar{t}}^* + (1-z_{\bar{t}}^*)\,x\right)^{\alpha}.$$

and z_t^* is defined in (22).

In case $r^E_{\varnothing}(G,\bar{t}) + r^E_{\varnothing}(B,\bar{t}) < 1$, the first part of Lemma 13 prescribes that $W^Q_{\bar{t}}(G,\bar{t}) = W^Q_{\bar{t}}(B,\bar{t}) = W^Q_{\bar{t}}(\varnothing)$. Analogously to the previous case, we can solve for $r^Q(G,\bar{t})$ and $r^Q(B,\bar{t})$:

$$\begin{aligned} r^Q(G,\bar{t}) &= \frac{M_G\left(r^E_{\varnothing}(G,\bar{t})\right)^{\frac{1}{\alpha}}}{M_G\left(r^E_{\varnothing}(G,\bar{t})\right)^{\frac{1}{\alpha}} + M_B\left(r^E_{\varnothing}(B,\bar{t})\right)^{\frac{1}{\alpha}} + \left(1-z_{\bar{t}}^*\right)\cdot(T-\bar{t}+\theta)^{\frac{1}{\alpha}}\cdot\left(1-r^E_{\varnothing}(G,\bar{t})-r^E_{\varnothing}(B,\bar{t})\right)}, \\ r^Q(B,\bar{t}) &= \frac{M_B\left(r^E_{\varnothing}(B,\bar{t})\right)^{\frac{1}{\alpha}}}{M_G\left(r^E_{\varnothing}(G,\bar{t})\right)^{\frac{1}{\alpha}} + M_B\left(r^E_{\varnothing}(B,\bar{t})\right)^{\frac{1}{\alpha}} + \left(1-z_{\bar{t}}^*\right)\cdot(T-\bar{t}+\theta)^{\frac{1}{\alpha}}\cdot\left(1-r^E_{\varnothing}(G,\bar{t})-r^E_{\varnothing}(B,\bar{t})\right)}, \end{aligned} \tag{34}$$

Note that the expressions in (33) are special cases of (34) when $r^E_{\varnothing}(G,\bar{t}) + r^E_{\varnothing}(B,\bar{t}) = 1$. Therefore, we can assume without loss that $r^Q(G,\bar{t})$ and $r^Q(B,\bar{t})$ are given by (34). All of the above proves that the solution exists for $t=\bar{t}$.

We establish existence for all $t\in\mathcal{S}\backslash\{\bar{t}\}$ proceeding by backward induction. Consider a system given by

$$W^Q_{t_{|\mathcal{S}|-1}}(G,t_{|\mathcal{S}|-1}) = W^Q_{t_{|\mathcal{S}|-1}}(B,t_{|\mathcal{S}|-1}) = W^Q_{t_{|\mathcal{S}|-1}}(G,\bar{t}). \tag{35}$$

We next show that this system of equations always has a solution. Consider the following auxiliary system. Suppose that $r^Q(G,t_{|\mathcal{S}|-1}) + r^Q(B,t_{|\mathcal{S}|-1}) = c$ for some $c>0$ and consider the equation given by the first equality in (35). Then if $r^Q(G,t_{|\mathcal{S}|-1})$ approaches zero, the LHS approaches $+\infty$ while the RHS approaches a constant.[34] Similarly, the RHS is strictly bigger than the LHS when $r^Q(G,t_{|\mathcal{S}|-1}) = c$. Moreover, the LHS is continuous and strictly decreasing in $r^Q(G,t_{|\mathcal{S}|-1})$, while the RHS is continuous and strictly increasing in $r^Q(G,t_{|\mathcal{S}|-1})$ given $r^Q(G,t_{|\mathcal{S}|-1}) + r^Q(B,t_{|\mathcal{S}|-1}) = c$. Therefore, by the Intermediate Value Theorem, for a given $c>0$, there exists a unique pair $r^Q(G,t_{|\mathcal{S}|-1}), r^Q(B,t_{|\mathcal{S}|-1})$ such that $r^Q(G,t_{|\mathcal{S}|-1}) + r^Q(B,t_{|\mathcal{S}|-1}) = c$ and $W^Q_{t_{|\mathcal{S}|-1}}(G,t_{|\mathcal{S}|-1}) = W^Q_{t_{|\mathcal{S}|-1}}(B,t_{|\mathcal{S}|-1})$. Also note that both $r^Q(G,t_{|\mathcal{S}|-1})$ and $r^Q(B,t_{|\mathcal{S}|-1})$ are then strictly increasing in c. Next, take some $c>0$ and assume that $r^Q(G,t_{|\mathcal{S}|-1}) + r^Q(B,t_{|\mathcal{S}|-1}) = c$. Consider the equality

$$W^Q_{t_{|\mathcal{S}|-1}}(G,t_{|\mathcal{S}|-1}) = W^Q_{t_{|\mathcal{S}|-1}}(G,\bar{t}) \tag{36}$$

as an equation in c. The RHS of (36) is continuous and strictly increasing in c, it approaches $+\infty$ when c approaches 1. As established before, the LHS of (36) is continuous and strictly decreasing in c, it approaches $+\infty$ when c approaches zero. Therefore, there exist unique $r^Q(G,t_{|\mathcal{S}|-1})$ and $r^Q(B,t_{|\mathcal{S}|-1})$ such that $W^Q_{t_{|\mathcal{S}|-1}}(G,t_{|\mathcal{S}|-1}) = W^Q_{t_{|\mathcal{S}|-1}}(B,t_{|\mathcal{S}|-1}) = W^Q_{t_{|\mathcal{S}|-1}}(G,\bar{t})$. Similarly, proceeding backward through support $\mathcal{S}$, we solve for the remaining $r^Q(G,t)$ and $r^Q(B,t)$, which completes the proof. □

The following lemma establishes that whenever $w(\beta)=\beta^{\alpha}$ and $w^c(\beta)=\theta\cdot\beta^{\alpha}$, the strategies that

[34] The claim is valid because both $w(\cdot)$ and $w^c(\cdot)$ are unbounded.

constitute a solution to either system (30) or system (31) are continuously differentiable in $r^E_\varnothing(G,\bar{t})$ and $r^E_\varnothing(B,\bar{t})$, including at $r^E_\varnothing(G,\bar{t})+r^E_\varnothing(B,\bar{t})=1$. The same is then true for all associated equilibrium objects $W^Q_t(m,\tau)$, β and $i(m,t)$, as they all are continuously differentiable functions of the strategies. The statement of this lemma is valid for any continuously differentiable $w(\cdot)$, $w^c(\cdot)$, but the statement for this particular functional form is enough for the needs of the paper and is significantly easier to prove.

Lemma 15. *Suppose $w(\beta)=\beta^\alpha$ and $w^c(\beta)=\theta\cdot\beta^\alpha$, and strategy profile $\{r^\gamma_\eta(m,t)\}$ solves either system (30) or system (31). Then $r^\gamma_\eta(m,t)$ is a continuously differentiable function of $r^E_\varnothing(G,\bar{t})$ and $r^E_\varnothing(B,\bar{t})$ for all $r^E_\varnothing(G,\bar{t})+r^E_\varnothing(B,\bar{t})\leqslant 1$.*

Proof. First note that $r^\gamma_\eta(m,t)$ exists by Lemma 14. Next, the strategy profile for the expert is fixed by the premise of Lemma 13 and, therefore, is trivially a continuously differentiable function of $r^E_\varnothing(G,\bar{t})$ and $r^E_\varnothing(B,\bar{t})$. Therefore, we are left to establish that $r^Q(m,t)$ is a continuously differentiable function of $r^E_\varnothing(G,\bar{t})$ and $r^E_\varnothing(B,\bar{t})$ for all $m\in\{G,B\}$ and all $t\in\mathcal{S}$. As expressions in (33) coincide with the ones in (34) for $r^E_\varnothing(G,\bar{t})+r^E_\varnothing(B,\bar{t})=1$, without loss, we restrict ourselves to the case $r^E_\varnothing(G,\bar{t})+r^E_\varnothing(B,\bar{t})<1$. Both expressions in (34) are continuously differentiable functions of $r^E_\varnothing(G,\bar{t})$ and $r^E_\varnothing(B,\bar{t})$ for $r^E_\varnothing(G,\bar{t})+r^E_\varnothing(B,\bar{t})\leqslant 1$. Thus, it is left to show the same for $r^Q(G,t)$ and $r^Q(B,t)$ for $t\in\mathcal{S}\backslash\{\bar{t}\}$. We proceed using backward induction. Consider two equalities

$$W^Q_{t_{|\mathcal{S}|-1}}\left(G,t_{|\mathcal{S}|-1}\right)=W^Q_{t_{|\mathcal{S}|-1}}\left(G,\bar{t}\right),$$
$$W^Q_{t_{|\mathcal{S}|-1}}\left(B,t_{|\mathcal{S}|-1}\right)=W^Q_{t_{|\mathcal{S}|-1}}\left(B,\bar{t}\right).$$

Given $r^Q(G,\bar{t})$ and $r^Q(B,\bar{t})$, they constitute a system of equations on $r^Q\left(G,t_{|\mathcal{S}|-1}\right)$ and $r^Q\left(B,t_{|\mathcal{S}|-1}\right)$. By writing out these equations explicitly using (12), (23), and (24), one can see that with $w(\beta)=\beta^\alpha$ and $w^c(\beta)=\theta\cdot\beta^\alpha$, all terms relating to $\beta_{t_{|\mathcal{S}|-2}}$ cancel out. As a result, both $r^Q\left(G,t_{|\mathcal{S}|-1}\right)$ and $r^Q\left(B,t_{|\mathcal{S}|-1}\right)$ depend only on future strategies $r^Q(G,\bar{t})$ and $r^Q(B,\bar{t})$, but not on past strategies $r^Q(G,t)$ and $r^Q(B,t)$ for $t\leqslant t_{|\mathcal{S}|-2}$. Therefore, by the Implicit Function Theorem, $r^Q\left(G,t_{|\mathcal{S}|-1}\right)$ and $r^Q\left(B,t_{|\mathcal{S}|-1}\right)$ are continuously differentiable functions of $r^Q(G,\bar{t})$ and $r^Q(B,\bar{t})$, which, in turn, implies that they are continuously differentiable functions of $r^E_\varnothing(G,\bar{t})$ and $r^E_\varnothing(B,\bar{t})$. Proceeding backwards, we establish the claim for all $r^Q(G,t)$ and $r^Q(B,t)$ for $t\in\mathcal{S}$. ☐

The two following lemmas are mostly technical and provide little intuition for the main problem.

Lemma 16. *Suppose*

$$m(x)=\left(\chi_1\left(a_1+bx\right)^\alpha+\ldots+\chi_k\left(a_k+bx\right)^\alpha\right)^{\frac{1}{\alpha}}-bx$$

for $k\geqslant 2$, $b>0$, $\sum_{i=1}^k\chi_i=1$, $a_1,\ldots,a_k\geqslant 0$ with $a_i,a_j>0$ for some $i,j\in\{1,\ldots,k\}$, $i\neq j$. Then $m(x)$ is strictly decreasing in $x>0$ when $\alpha>1$ and is strictly increasing when $0<\alpha<1$.

Proof. Begin by observing that

$$\frac{1}{b}\cdot\frac{dm(x)}{dx}=\frac{\chi_1\left(a_1+bx\right)^{\alpha-1}+\ldots+\chi_k\left(a_k+bx\right)^{\alpha-1}}{\left(\chi_1\left(a_1+bx\right)^\alpha+\ldots+\chi_k\left(a_k+bx\right)^\alpha\right)^{\frac{\alpha-1}{\alpha}}}-1.$$

If $0<\alpha<1$, then $f(x):=x^{\frac{\alpha-1}{\alpha}}$ is strictly convex for $x>0$, so (since $\sum_{i=1}^k\chi_i=1$)

$$\chi_1\left(a_1+bx\right)^{\alpha-1}+\ldots+\chi_k\left(a_k+bx\right)^{\alpha-1}>\left(\chi_1\left(a_1+bx\right)^\alpha+\ldots+\chi_k\left(a_k+bx\right)^\alpha\right)^{\frac{\alpha-1}{\alpha}}. \tag{37}$$

Therefore, $\frac{dm(x)}{dx} > 0$ for $x > 0$ if $\alpha < 1$. Conversely, if $\alpha > 1$, then $f(x) = x^{\frac{\alpha-1}{\alpha}}$ is strictly concave for $x > 0$, so the converse to (37) holds, and $\frac{dm(x)}{dx} < 0$ for $x > 0$. □

Lemma 17. *Suppose $w(\beta) = \beta^\alpha$ and $w^c(\beta) = \theta \cdot \beta^\alpha$, and the strategy profile $\{r^\gamma_\eta(m,t)\}$ solves either system* (30)*, or system* (31) *given some $\left[\mathcal{S}, r^E_\varnothing(G,\bar{t}), r^E_\varnothing(B,\bar{t})\right]$. Then for any $m \in \{G,B\}$:*

$$\frac{\partial}{\partial r^E_\varnothing(m,\bar{t})}\left(\frac{\left(1-z^*_{\bar{t}}\right)\cdot\left(1-r^E_\varnothing(G,\bar{t})-r^E_\varnothing(B,\bar{t})\right)}{1-r^Q(G,\bar{t})-r^Q(B,\bar{t})}\right) > 0 \text{ if } \alpha < 1;$$

$$\frac{\partial}{\partial r^E_\varnothing(m,\bar{t})}\left(\frac{\left(1-z^*_{\bar{t}}\right)\cdot\left(1-r^E_\varnothing(G,\bar{t})-r^E_\varnothing(B,\bar{t})\right)}{1-r^Q(G,\bar{t})-r^Q(B,\bar{t})}\right) < 0 \text{ if } \alpha > 1.$$

Proof. Lemmas 14 and 15 imply that $r^Q(m,t)$ exists for all $m \in \{G,B\}$ and $t \in \mathcal{S}$ and is continuously differentiable in $r^E_\varnothing(G,\bar{t})$ and $r^E_\varnothing(B,\bar{t})$. Next, from (34) we can calculate that

$$\frac{\left(1-z^*_{\bar{t}}\right)\cdot\left(1-r^E_\varnothing(G,\bar{t})-r^E_\varnothing(G,\bar{t})\right)}{1-r^Q(G,\bar{t})-r^Q(B,\bar{t})} = \frac{M_G\left(r^E_\varnothing(G,\bar{t})\right)^{\frac{1}{\alpha}}+M_B\left(r^E_\varnothing(B,\bar{t})\right)^{\frac{1}{\alpha}}}{(T-\bar{t}+\theta)^{\frac{1}{\alpha}}} + (1-z^*_{\bar{t}})\cdot\left(1-r^E_\varnothing(G,\bar{t})-r^E_\varnothing(B,\bar{t})\right),$$

and see that the RHS of the expression above is the sum of two functions, each of which satisfy the requirements of Lemma 16, and a constant. Therefore, the statement of the lemma follows directly from Lemma 16. □

Lemma 18. *Suppose $w(\beta) = \beta^\alpha$ and $w^c(\beta) = \theta \cdot \beta^\alpha$, and strategy profile $\{r^\gamma_\eta(m,t)\}$ solves either system* (30)*, or system* (31) *given some $\left[\mathcal{S}, r^E_\varnothing(G,\bar{t}), r^E_\varnothing(B,\bar{t})\right]$. Then for any $m \in \{G,B\}$ and any $t \in \mathcal{S}$, $W^Q_{t_1}(m,t)$ is strictly increasing in $r^E_\varnothing(m,\bar{t})$ if $\alpha < 1$, and is strictly decreasing if $\alpha > 1$.*

Proof. By Lemma 15 we can assume without loss that making no report is on path at $\bar{t}$. Fix some $\mathcal{S} \subseteq \mathcal{T}$ and strategy profiles $\{r^\gamma_\eta(m,t)\}, \{\tilde{r}^\gamma_\eta(m,t)\}$ such that for some $m \in \{G,B\}$: $\tilde{r}^E_\varnothing(m,\bar{t}) > r^E_\varnothing(m,\bar{t})$, $\tilde{r}^E_\varnothing(-m,\bar{t}) = r^E_\varnothing(-m,\bar{t})$, and the remainders of $\{r^\gamma_\eta(m,t)\}$ and $\{\tilde{r}^\gamma_\eta(m,t)\}$ solve system (30) for $\left[\mathcal{S}, r^E_\varnothing(G,\bar{t}), r^E_\varnothing(B,\bar{t})\right]$ and $\left[\mathcal{S}, \tilde{r}^E_\varnothing(G,\bar{t}), \tilde{r}^E_\varnothing(B,\bar{t})\right]$, respectively. Let β and $\tilde{\beta}$ denote the respective beliefs corresponding to the two strategy profiles, and by $W^Q_t(m,\tau)$ and $\tilde{W}^Q_t(m,\tau)$ the respective quack's values from reports.

What follows is the proof for $\alpha < 1$ (case $\alpha > 1$ is analogous). Assume, by way of contradiction, that $\tilde{W}^Q_{t_1}(m,t_1) \leqslant W^Q_{t_1}(m,t_1)$. Then $\tilde{r}^Q(m,t_1) \geqslant r^Q(m,t_1)$ for $m \in \{G,B\}$. From (25) we then get that $\tilde{\beta}_{t_1} \geqslant \beta_{t_1}$. This, in turn, implies that $\tilde{r}^Q(G,t_2) \geqslant r^Q(G,t_2)$ and $\tilde{r}^Q(B,t_2) \geqslant r^Q(B,t_2)$ because $W^Q_{t_1}(m,t_1) = W^Q_{t_1}(m,t_2)$ and $\tilde{W}^Q_{t_1}(m,t_1) = \tilde{W}^Q_{t_1}(m,t_2)$ for $m \in \{G,B\}$. Iterating this logic further, we get $\tilde{\beta}_{t_{|\mathcal{S}-1|}} \geqslant \beta_{t_{|\mathcal{S}-1|}}$. From Lemma 17 we know that

$$\tilde{\beta}^\omega(\varnothing) = \tilde{\beta}_{\bar{t}} = \tilde{\beta}_{t_{|\mathcal{S}-1|}} \cdot \frac{(1-z^*_{\bar{t}})\cdot\left(1-\tilde{r}^E_\varnothing(G,\bar{t})-\tilde{r}^E_\varnothing(G,\bar{t})\right)}{1-\tilde{r}^Q(G,\bar{t})-\tilde{r}^Q(B,\bar{t})} > \beta_{t_{|\mathcal{S}-1|}} \cdot \frac{(1-z^*_{\bar{t}})\cdot\left(1-r^E_\varnothing(G,\bar{t})-r^E_\varnothing(G,\bar{t})\right)}{1-r^Q(G,\bar{t})-r^Q(B,\bar{t})} = \beta_{\bar{t}} = \beta^\omega(\varnothing),$$

and, therefore, $\tilde{W}^Q_{t_1}(\varnothing) > W^Q_{t_1}(\varnothing)$. This gives us a contradiction with the initial assumption $\tilde{W}^Q_{t_1}(m,t_1) \leqslant W^Q_{t_1}(m,t_1)$ because we must have $W^Q_{t_1}(\varnothing) = W^Q_{t_1}(G,t_1)$ and $\tilde{W}^Q_{t_1}(\varnothing) = \tilde{W}^Q_{t_1}(G,t_1)$. □

Proof of Proposition 4 (applies also under the conditions stated in Proposition 8).

Part 1. For any set of parameters $\left[\mathcal{S}, r^E_\varnothing(G,\bar{t}), r^E_\varnothing(B,\bar{t})\right]$ with $|\mathcal{S}| = 1$, $r^E_\varnothing(G,\bar{t}) > 0$, and $r^E_\varnothing(B,\bar{t}) > 0$, Proposition 2 and $r^E_\varnothing(m,\bar{t})$ pin down the expert's strategy. We next show that there exist $r^Q(G,\bar{t})$ and $r^Q(B,\bar{t})$ such that conditions in Lemma 13 are satisfied, and hence an informative equilibrium exists, which proves the first statement of the proposition.

The first condition one needs to verify in order to establish the existence is $W^Q_{\bar{t}}(G,\bar{t}) = W^Q_{\bar{t}}(B,\bar{t})$, which can be written as

$$(T-\bar{t})\cdot w\left(\beta(G,\bar{t})\right) + p_0\cdot w^c\left(\beta^G(G,\bar{t})\right) + (1-p_0)\cdot w^c\left(\beta^B(G,\bar{t})\right) = \\ = (T-\bar{t})\cdot w\left(\beta(B,\bar{t})\right) + p_0\cdot w^c\left(\beta^G(B,\bar{t})\right) + (1-p_0)\cdot w^c\left(\beta^B(B,\bar{t})\right). \quad (38)$$

Suppose first that $r^E_\varnothing(G,\bar{t}) + r^E_\varnothing(B,\bar{t}) = 1$. Then from Proposition 1, $r^Q(G,\bar{t}) + r^Q(B,\bar{t}) = 1$. From (23) and (24) we see that the LHS is strictly decreasing in $r^Q(G,\bar{t})$, and the RHS is strictly increasing in $r^Q(G,\bar{t})$ (conditional on $r^Q(G,\bar{t}) + r^Q(B,\bar{t}) = 1$). Moreover, all six beliefs in (38) are always positive irrespective of $r^Q(G,\bar{t})$ because $r^E_\varnothing(G,\bar{t}) > 0$ and $r^E_\varnothing(B,\bar{t}) > 0$. Therefore, when $r^Q(G,\bar{t}) = 0$ the LHS strictly dominates the RHS, and when $r^Q(G,\bar{t}) = 1$ the RHS strictly dominates the LHS. By the Intermediate Value Theorem, there exists a unique $r^Q(G,\bar{t})$ (and, therefore, a unique $r^Q(B,\bar{t}) = 1 - r^Q(G,\bar{t})$ due to (31)) such that (38) is satisfied.

In case $r^E_\varnothing(G,\bar{t}) + r^E_\varnothing(B,\bar{t}) < 1$, we further need to ensure that $W^Q_{\bar{t}}(G,\bar{t}) = W^Q_{\bar{t}}(\varnothing)$, i.e., that the value of not making a report at $\bar{t}$ is equal to the value of making a report:

$$(T-\bar{t})\cdot w\left(\beta(G,\bar{t})\right) + p_0\cdot w^c\left(\beta^G(G,\bar{t})\right) + (1-p_0)\cdot w^c\left(\beta^B(G,\bar{t})\right) = (T-\bar{t})\cdot w\left(\beta_{\bar{t}}\right) + w^c\left(\beta_{\bar{t}}\right). \quad (39)$$

By the same logic as above, we know that for any given $c > 0$ there exist unique $r^Q(G,\bar{t})$ and $r^Q(B,\bar{t})$ such that (38) is satisfied and $r^Q(G,\bar{t}) + r^Q(B,\bar{t}) = c$. Further, note that $r^Q(G,\bar{t})$ and $r^Q(B,\bar{t})$ that are obtained as a solution to this auxiliary system are both increasing in c.[35] Finally, consider (39) as an equation in c. From the previous observation it follows that its LHS is decreasing in c, while the RHS is increasing in c. If $c = 0$ then the LHS dominates the RHS, and if $c = 1$ the RHS dominates the LHS. Therefore, by the Intermediate Value Theorem, there exists a unique c such that (39) is satisfied. Solving (38) using this c gives $r^Q(G,\bar{t})$ and $r^Q(B,\bar{t})$ that uniquely solve the original system of (38) and (39).

Part 2. The first part has established the existence of equilibrium with any $|\mathcal{S}| = 1$. We next construct a relay reticent equilibrium for a given $\mathcal{S}$ with $|\mathcal{S}| \geqslant 2$ and $r^E_\varnothing(G,\bar{t}) = r^E_\varnothing(B,\bar{t}) = 0$. Since states are symmetric ($p_0 = \frac{1}{2}$) and so is the expert's strategy, given any $t \in \mathcal{S}$, for the quack to be indifferent between reports (G,t) and (B,t), it must be that $r^Q(G,t) = r^Q(B,t)$. Thus, we are only left to ensure that the quack's indifference between a report, say, report G, and no report can be satisfied, and to verify that the constructed equilibrium is incentive compatible for the informed expert. The fact that the equilibrium is assumed to be a relay one fixes the expert's strategy and the values of z^*_t. Therefore, values $r^E(G,t) = r^E(B,t) = \frac{1}{2}\cdot z^*_t$ are pinned down as well. By the definition of z^*_t in (22), therefore, $r^E(G,t) = r^E(B,t) < \frac{1}{2}$ for any t. Then $W^Q_{t_1}(G,t_1)$ can be expressed as

$$(T-t_1)\cdot w\left(\beta_0\cdot\frac{r^E(G,t_1)}{r^Q(G,t_1)}\right) + p_0\cdot w^c\left(\beta_0\cdot\frac{\pi}{p^*_0}\cdot\frac{r^E(G,t_1)}{r^Q(G,t_1)}\right) + (1-p_0)\cdot w^c\left(\beta_0\cdot\frac{1-\pi}{p^*_0}\cdot\frac{r^E(G,t_1)}{r^Q(G,t_1)}\right).$$

From the expression above we see that $W^Q_{t_1}(G,t_1)$ is fully determined by $r^Q(G,t_1)$ and is decreasing in $r^Q(G,t_1)$. At the same time, larger $r^Q(G,t_1)$ (and $r^Q(B,t_1) = r^Q(G,t_1)$) imply higher β_{t_1}. Because the quack must be indifferent between reports at t_1 and t_2, higher β_{t_1} implies larger $r^Q(G,t_2)$ (and $r^Q(B,t_2) = r^Q(G,t_2)$). Proceeding by induction, we get that $W^Q_{t_1}(\varnothing)$ is strictly increasing in $r^Q(G,t_1)$.

When $r^Q(G,t_1) = r^E(G,t_1)$, we have that $\beta_t = \beta_0$ for all $t \in \mathcal{T}$ (remember that $r^E_\varnothing(G,\bar{t}) = r^E_\varnothing(B,\bar{t}) = 0$

[35] At least one of $r^E_\varnothing(G,\bar{t})$ and $r^E_\varnothing(B,\bar{t})$ must be higher for a higher c, and (38) implies that higher $r^E_\varnothing(G,\bar{t})$ implies higher $r^E_\varnothing(B,\bar{t})$.

in a relay equilibrium, so following the logic from Proposition 6, this claim extends to the Godwin point as well). Therefore, $W_{t_1}^Q(\varnothing) = (T - t_1) \cdot w(\beta_0) + w^c(\beta_0)$. In this case, because $w^c(\cdot)$ is convex, we have

$$(T - t_1) \cdot w(\beta_0) + p_0 \cdot w^c\left(\beta_0 \cdot \frac{\pi}{p_0^*}\right) + (1 - p_0) \cdot w^c\left(\beta_0 \cdot \frac{1 - \pi}{p_0^*}\right) \geqslant (T - t_1) \cdot w(\beta_0) + w^c(\beta_0),$$

i.e., the value of making a report is weakly higher then the value of staying silent. At the same time, when $r^Q(G, t_1) + r^Q(B, t_1) \to 1$, the opposite is true: the value of making no report strictly dominates the value of making a report. Therefore, by the Intermediate Value Theorem, there exists a unique value for $r^Q(G, t_1)$ and $r^Q(B, t_1)$ such that the quack is indifferent between making a report at any period and not making a report. Finally, because at such a solution we have $r^Q(m, t_1) \geqslant r^E(m, t_1)$ for both m, this implies $\beta(m, t_1) \leqslant \beta_0 \leqslant \beta_{t_1}$, which satisfies Theorem 1/Proposition 6. From their proof for convex $w^c(\cdot)$, we know that $\beta^m(m, t)$ – and, consequently, $\Delta w_\eta(m, t)$ for $m = \eta$ – are decreasing on $\mathcal{S}$, which proves that $r_\eta^E(\eta, t) = 1$ and $r_\varnothing^E(m, t) = 0$ for $\eta, m \in \{G, B\}$ is an optimal strategy for the expert, i.e., the constructed profile does indeed constitute an equilibrium.

Part 3.1. We first establish the third claim for relay equilibria. Note that the conditions in this statement directly contradict both conditions in Proposition 8, so Proposition 8 does not apply to the third statement in Proposition 4. Therefore, we only need to prove it for $\pi = 1$.

Assume by way of contradiction that there exists an informative relay equilibrium on some support $\mathcal{S}$ with $|\mathcal{S}| \geqslant 3$ with the respective strategy profile $\{r_\eta^\gamma(m, t)\}$, belief β, and the quack's value function $W_t^Q(m, \tau)$. Then there must exist another relay equilibrium with the same $\mathcal{S}$ and $\tilde{r}_\varnothing^E(G, t) = \tilde{r}_\varnothing^E(B, t) = 0$. In particular, by Lemma 14, there exists a strategy profile $\{\tilde{r}_\eta^\gamma(m, t)\}$ which solves system (30) given these parameters. Denote the belief profile associated with it as $\tilde{\beta}$ and the respective quack's value as $\tilde{W}_t^Q(m, \tau)$. To confirm that this profile constitutes a relay equilibrium on $\mathcal{S}$, the only condition that needs to be verified is the informed expert's incentive compatibility. By the proof of Theorem 1, for $\mathcal{S}$ with $|\mathcal{S}| \geqslant 3$, this is equivalent to verifying that $\tilde{\beta}_{t_1} \geqslant \tilde{\beta}(m, t_1)$ for $m \in \{G, B\}$, because $\tilde{r}_\varnothing^E(G, t) = \tilde{r}_\varnothing^E(B, t) = 0$. Theorem 1 also implies that in the original equilibrium we have $\beta_{t_1} \geqslant \beta(m, t_1)$. By Lemma 18, because $r_\varnothing^E(m, t) \geqslant \tilde{r}_\varnothing^E(m, t) = 0$ for $m \in \{G, B\}$ and $\alpha < 1$, we have $W_{t_1}^Q(m, t_1) \geqslant \tilde{W}_{t_1}^Q(m, t_1)$. This implies that $r^Q(m, t_1) \leqslant \tilde{r}^Q(m, t_1)$ for $m \in \{G, B\}$ and, therefore, $\tilde{\beta}_{t_1} \geqslant \beta_{t_1} \geqslant \beta(m, t_1) \geqslant \tilde{\beta}(m, t_1)$ for $m \in \{G, B\}$, which completes the argument.

We have established that if there exists a relay equilibrium on $\mathcal{S}$, then there must also exist a relay equilibrium with the same support and $\tilde{r}_\varnothing^E(G, t) = \tilde{r}_\varnothing^E(B, t) = 0$. By Theorem 1, there exists $m \in \{G, B\}$ such that $\tilde{\beta}(m, t_1) \leqslant \beta_0$, and therefore (remember that $\pi = 1$, so $p_0^* = p_0$):

$$\tilde{W}_{t_1}^Q(m, t_1) \leqslant (T - t_1) \cdot w(\beta_0) + p_0 \cdot w^c\left(\frac{\beta_0}{p_0}\right). \tag{40}$$

At the same time, because in such an equilibrium $\beta_t \geqslant \beta_0$ for all $t \in \mathcal{S}$ (again by Proposition 8), we have

$$\tilde{W}_{t_1}^Q(\varnothing) \geqslant (T - t_1) \cdot w(\beta_0) + w^c(\beta_0). \tag{41}$$

Finally, $\tilde{W}_{t_1}^Q(m, t_1) = \tilde{W}_{t_1}^Q(\varnothing)$ implies

$$p_0 \cdot w^c\left(\frac{\beta_0}{p_0}\right) \geqslant w^c(\beta_0). \tag{42}$$

Since $w^c(0) = 0$, if $w^c(\cdot)$ is strictly concave, then (42) can not be satisfied. This gives us a contradiction, meaning no relay equilibrium $\{r_\eta^\gamma(m, t)\}$ exists.

Part 3.2. We next establish the third claim of the proposition for delay equilibria. As above, it is enough to show it for $\pi = 1$. Consider a delay equilibrium with strategy profile $\{r^\gamma_\eta(m,t)\}$. Remember from Theorem 1/Proposition 8 that in any equilibrium with $|\mathcal{S}| \geqslant 3$, it must be that $\beta(m,t_1) \leqslant \beta_{t_1}$. We first show that $r^E_G(G, t_{|\mathcal{S}|-1}) = r^E_B(B, t_{|\mathcal{S}|-1}) < 1$.[36] Assume the contrary, i.e., $r^E_G(G, t_{|\mathcal{S}|-1}) = r^E_B(B, t_{|\mathcal{S}|-1}) = 1$. Then there are two possible cases: either $\beta(m,t_1) = \beta_{t_1}$ for all $m \in \{G,B\}$, or $\beta(m,t_1) < \beta_{t_1}$ for some $m \in \{G,B\}$.

If $\beta(m,t_1) = \beta_{t_1}$ for both $m \in \{G,B\}$, then to sustain the quack's indifference between the reports within the support, we should have $\beta(m,t) = \beta_t = \beta_0$ for all $t \leqslant t_{|\mathcal{S}|-1}$. Consider a modified strategy profile $\{\hat{r}^\gamma_\eta(m,t)\}$ with $\hat{r}^E_G(G,t) = \hat{r}^E_B(B,t) = 1$ and $\hat{r}^Q(G,t) = p^*_0 \cdot \hat{z}^*_t$, $\hat{r}^Q(B,t) = (1-p^*_0)\cdot \hat{z}^*_t$ for all $t \in \mathcal{S}\backslash\{\bar{t}\}$ and $\hat{r}^\gamma_\eta(m,t) = r^\gamma_\eta(m,t)$ for all γ, η, m and $t \notin \mathcal{S}\backslash\{\bar{t}\}$.[37] The two strategy profiles generate the same beliefs. Indeed, due to (23), in the modified profile we still have $\hat{\beta}(m,t) = \hat{\beta}_t = \beta_0$ for $t \leqslant t_{|\mathcal{S}|-1}$. At the same time, because in both the original profile and the modified profile we have $r^E_G(G,t_{|\mathcal{S}|-1}) = r^E_B(B,t_{|\mathcal{S}|-1}) = 1$ and $\hat{r}^E_G(G,t_{|\mathcal{S}|-1}) = \hat{r}^E_B(B,t_{|\mathcal{S}|-1}) = 1$, it follows that $z^*_{\bar{t}} = \hat{z}^*_{\bar{t}}$ due to (14). Because the strategies at $\bar{t}$ are the same in either profile as well, beliefs after any action at $\bar{t}$ coincide between $\{r^\gamma_\eta(m,t)\}$ and $\{\hat{r}^\gamma_\eta(m,t)\}$. Therefore, if the original profile constituted an equilibrium, the modified profile constitutes a *relay* equilibrium, which, as we proved above, is impossible.

Finally, consider the case when $\beta(m,t_1) < \beta_{t_1}$ for some $m \in \{G,B\}$. Then, since $|\mathcal{S}| \geqslant 3$, in order to sustain the quack's indifference between reports, β_t should be strictly increasing for $t \leqslant t_{|\mathcal{S}|-1}$ (see Step 3 in the proof of Theorem 1). Because the equilibrium was assumed to be of delay type, there exists $t_i \in \mathcal{S}$ such that $r^E_G(G,t_i) = r^E_B(B,t_i) < 1$, i.e., the expert is indifferent between reporting $m \in \{G,B\}$ at t_i and t_{i+1} such that $t_{i+1} < \bar{t}$. The quack is also indifferent between either report at t_i and t_{i+1}, so for both $m \in \{G,B\}$:

$$w^c(\beta^G(m,t_i)) - w^c(\beta^B(m,t_i)) = w^c(\beta^G(m,t_{i+1})) - w^c(\beta^B(m,t_{i+1})). \tag{43}$$

Since we only consider the case $\pi = 1$, we have $\beta^G(B,t) = \beta^B(G,t) = 0$ for any $t \in \mathcal{S}\backslash\{\bar{t}\}$. Because $w^c(\cdot)$ is a strictly increasing function, (43) directly implies that $\beta^G(m,t_i) = \beta^G(m,t_{i+1})$ and $\beta^B(m,t_i) = \beta^B(m,t_{i+1})$. This, in turn, implies that $\beta(m,t_i) = \beta(m,t_{i+1})$. Since β_t must be strictly increasing for $t \leqslant t_{|\mathcal{S}|-1}$, we have $\beta_{t_i} > \beta(m,t_i)$ for at least one m, but then report (m,t_{i+1}) is strictly better for the quack than (m,t_i) – a contradiction.

All in all, the preceding argument proves that $r^E_G(G,t_{|\mathcal{S}|-1}) = r^E_B(B,t_{|\mathcal{S}|-1}) < 1$, i.e., the expert with any $\eta \in \{G,B\}$ should be indifferent between making report $m = \eta$ in period $t_{|\mathcal{S}|-1}$ and in period $\bar{t}$. Moreover, the quack is also indifferent between these reports. The two indifferences (taking w.l.o.g. $m = \eta = G$) imply that

$$w^c(\beta^G(G,t_{|\mathcal{S}|-1})) - w^c(\beta^B(G,t_{|\mathcal{S}|-1})) = w^c(\beta^G(G,\bar{t})) - w^c(\beta^B(G,\bar{t})).$$

Again, since $\pi = 1$, $\beta^B(G,t_{|\mathcal{S}|-1}) = 0$. Since $w^c(\cdot)$ is strictly increasing and $\beta^B(G,\bar{t}) \geqslant 0$, it must be that $\beta^G(G,t_{|\mathcal{S}|-1}) \leqslant \beta^G(G,\bar{t})$. Strict inequality, however, is impossible, since then report $(G,\bar{t})$ would be strictly better for the quack than report $(G,t_{|\mathcal{S}|-1})$.[38] Therefore, it must be that $\beta^G(G,t_{|\mathcal{S}|-1}) = \beta^G(G,\bar{t})$ and $\beta^B(G,t_{|\mathcal{S}|-1}) = \beta^B(G,\bar{t}) = 0$, which together also imply that $\beta(G,t_{|\mathcal{S}|-1}) = \beta(G,\bar{t})$ by (17). Analogously, we can establish that $\beta(B,t_{|\mathcal{S}|-1}) = \beta(B,\bar{t})$. Because $\beta_{t_{|\mathcal{S}|-1}} \geqslant \beta(m,t_{|\mathcal{S}|-1})$ for both m, we get in the end that $\beta_{t_{|\mathcal{S}|-1}} \geqslant \beta(m,\bar{t})$ for both m. Further, $\beta^B(G,\bar{t}) = 0$ directly implies $r^E_\varnothing(G,\bar{t}) = 0$, and by the mirror

[36] The two reporting probabilities must be equal due to (SY).

[37] Remember that z^*_t depends on the expert's strategy. Here $\hat{z}^*_t$ are derived from profile $\{\hat{r}^\gamma_\eta(m,t)\}$. Note also that strategies in period t only depend on $\hat{z}^*_t$, which, in turn, only depends on the strategies up to $t-1$, so both can be constructed by induction on t. Together with the fact that $\hat{z}^*_t \in [0,1)$ this means that the strategies are well defined.

[38] $\beta(G,t_{|\mathcal{S}|-1}) \leqslant \beta_{t_{|\mathcal{S}|-1}}$, $\beta(G,t_{|\mathcal{S}|-1}) < \beta(G,\bar{t})$ and $\beta^G(G,t_{|\mathcal{S}|-1}) < \beta^G(G,\bar{t})$, $\beta^B(G,t_{|\mathcal{S}|-1}) \leqslant \beta^B(G,\bar{t})$.

logic, $r^E_\varnothing(B,\bar t)=0$, meaning that staying silent has to be on path at $\bar t$.

Finally, by the martingale property of beliefs, $\beta(m,\bar t)\leqslant\beta_{\bar t}$ for both m implies that $\beta_{\bar t}\geqslant\beta_{t_{|\mathcal{S}|-1}}$, meaning that β_t is weakly increasing on all of $\mathcal{S}$, and so (41) holds. At the same time, because $\beta(m,t_1)\leqslant\beta_{t_1}$ for at least one m, (40) holds as well. Therefore, (42) gives us a contradiction, same as for relay equilibrium. □

Proof of Proposition 5 (applies also under the conditions stated in Proposition 8).

Denote by $W^Q_t(m,\tau)$ and $\tilde W^Q_t(m,\tau)$ the quack's respective values of making a report and by β and $\tilde\beta$ the beliefs corresponding to strategy profiles $\{r^\gamma_\eta(m,t)\}$ and $\{\tilde r^\gamma_\eta(m,t)\}$. We first establish the statement of the proposition for relay equilibria.

We begin by showing that $W^Q_{t_1}(m,t_1)\leqslant\tilde W^Q_{t_1}(m,t_1)$ for both $m\in\{G,B\}$. Assume the contrary: that $W^Q_{t_1}(m,t_1)>\tilde W^Q_{t_1}(m,t_1)$ for both $m\in\{G,B\}$.[39] This directly implies $r^Q(m,t_1)<\tilde r^Q(m,t_1)$ for $m\in\{G,B\}$. From (25) it then follows that $\beta_{t_1}<\tilde\beta_{t_1}$. This, in turn, implies that $r^Q(G,t_2)<\tilde r^Q(G,t_2)$ and $r^Q(B,t_2)<\tilde r^Q(B,t_2)$, because $W^Q_{t_1}(m,t_1)=W^Q_{t_1}(m,t_2)$ and $\tilde W^Q_{t_1}(m,t_1)=\tilde W^Q_{t_1}(m,t_2)$ for $m\in\{G,B\}$. Iterating this logic further, we obtain that $\beta_t<\tilde\beta_t$ for all $t\in\mathcal{S}$. Additionally, by Theorem 1, we have $\beta_{t_k}\leqslant\beta_{t_{k+1}}\leqslant\ldots\leqslant\beta_{t_{k+n}}$ (we can extend the argument to $\beta_{t_{k+n}}$ because $\tilde r^E_\varnothing(m,t_{k+n})=0$ for $m\in\{G,B\}$). Therefore, $W^Q_{t_1}(\varnothing)\leqslant\tilde W^Q_{t_1}(\varnothing)$. Making no report is on path in both equilibria, thus $W^Q_{t_1}(G,t_1)=W^Q_{t_1}(\varnothing)$ and $\tilde W^Q_{t_1}(G,t_1)=\tilde W^Q_{t_1}(\varnothing)$. Consequently, $W^Q_{t_1}(m,t_1)\leqslant\tilde W^Q_{t_1}(m,t_1)$, which gives us a contradiction with the initial assumption.

Condition $W^Q_{t_1}(m,t)\leqslant\tilde W^Q_{t_1}(m,t)$ directly implies that $r^Q(m,t_1)\geqslant\tilde r^Q(m,t_1)$ – since in a relay equilibrium the strategy of the expert is fixed, – and, therefore, that $r^Q(m,t)\geqslant\tilde r^Q(m,t)$ and $|i(m,t)|\leqslant|\tilde i(m,t)|$ for all $t\in\mathcal{S}$. Finally, $|i(m,t)|=0$ for any $t\notin\mathcal{S}$, meaning that $|i(m,t)|<|\tilde i(m,t)|$ for $t\in\tilde{\mathcal{S}}\backslash\mathcal{S}$.

To extend the proof to delay equilibria, we next show that for any reticent delay equilibrium, we can construct a relay equilibrium with the same beliefs after any history, hence the argument above applies. From Theorem 1/Proposition 8 in any reticent equilibrium with $|\mathcal{S}|\geqslant 2$, it must be that $\beta(m,t_1)\leqslant\beta_{t_1}$. There are two possible cases: either $\beta(m,t_1)=\beta_{t_1}$ for all $m\in\{G,B\}$, or $\beta(m,t_1)<\beta_{t_1}$ for some $m\in\{G,B\}$.

Consider the case when $\beta(m,t_1)<\beta_{t_1}$ for some $m\in\{G,B\}$. Then, since $|\mathcal{S}|\geqslant 2$, in order to sustain the quack's indifference between reports, $\beta^m(m,t)$ should be strictly decreasing for $t\in\mathcal{S}$. Therefore, $\Delta w_\eta(m,t)$ is strictly decreasing as well (see Steps 2 and 3 in the proof of Theorem 1). Step 1 of Theorem 1/Proposition 8 then implies that $r^E_m(m,t)=1$ for all $t\in\mathcal{S}$. From (SY) we get that $r^E_G(G,t)=r^E_B(B,t)=1$, and so the equilibrium is a relay one, which contradicts the assumption that we are dealing with a delay one.

If $\beta(m,t_1)=\beta_{t_1}$ for both $m\in\{G,B\}$, then to sustain the quack's indifference between the reports within the support, we should have $\beta(m,t)=\beta_t=\beta_0$ for all $t\in\mathcal{S}$. Consider a modified strategy profile $\{\hat r^\gamma_\eta(m,t)\}$ with $\hat r^E_G(G,t)=\hat r^E_B(B,t)=1$ and $\hat r^Q(G,t)=p^*_0\cdot\hat z^*_t$, $\hat r^Q(B,t)=(1-p^*_0)\cdot\hat z^*_t$ for all $t\in\mathcal{S}\backslash\{\bar t\}$ and $\hat r^\gamma_\eta(m,t)=r^\gamma_\eta(m,t)$ for all γ,η,m and $t\notin\mathcal{S}\backslash\{\bar t\}$.[40] The original and the modified strategy profiles generate the same beliefs after any history. Indeed, due to (23), in the modified profile we still have $\hat\beta(m,t)=\hat\beta_t=\beta_0$ for $t\in\mathcal{S}$. Therefore, if the original profile constituted a (delay) equilibrium, the new profile constitutes a (relay) equilibrium as well. This completes the proof. □

[39]Because $W^Q_{t_1}(G,t_1)=W^Q_{t_1}(B,t_1)$ and $\tilde W^Q_{t_1}(G,t_1)=\tilde W^Q_{t_1}(B,t_1)$, we have that either $W^Q_{t_1}(m,t_1)>\tilde W^Q_{t_1}(m,t_1)$ for both $m\in\{G,B\}$, or $W^Q_{t_1}(m,t_1)<\tilde W^Q_{t_1}(m,t_1)$ for both $m\in\{G,B\}$.

[40]Remember that z^*_t depends on the expert's strategy. Here $\hat z^*_t$ are derived from profile $\{\hat r^\gamma_\eta(m,t)\}$. Note also that strategies in period t only depend on $\hat z^*_t$, which, in turn, only depends on the strategies up to $t-1$, so both can be constructed by induction on t. Together with the fact that $\hat z^*_t\in[0,1)$ this means that the strategies are well defined.

Proofs for Section 6

Proof of Proposition 7. In an ideal equilibrium, $r^E_\eta(\eta,t) = 1$ and $r^E_\varnothing(m,t) = r^Q(m,t) = 0$ for all $\eta, m \in \{G,B\}$ and $t \in \mathcal{S}$. The prescribed strategy is trivially optimal for the informed expert, since it yields the highest possible payoff for the remainder of the game with probability one: he gets $\bar{w} := \lim_{x\to+\infty} w(x)$ until T and $\bar{w}^c := \lim_{x\to+\infty} w^c(x)$ at T. The uninformed expert's preference for staying silent at any t is at least as strong as that of the quack (due to the option value of receiving news in the future and obtaining the maximal continuation payoff): formally, if $W^Q_{t_1}(\varnothing) \geqslant W^Q_{t_1}(m,t_1)$, then $V^E_{t_1,\varnothing} \geqslant W^E_{t_1,\varnothing}(m,t_1)$. Therefore, a necessary and sufficient condition for an ideal equilibrium to exist is for it to be optimal for the quack to stay silent at all $t \in \mathcal{S}$.

Since the analyst's reputation jumps to $b(m,t) = 1$ after any report in an ideal equilibrium, due to the martingale property of beliefs, it must decrease after no report. Therefore, any delay is costly for the quack, so it is sufficient to verify that reporting at t_1 is not optimal for him. The quack's payoffs generated by reports (G,t_1) and (B,t_1) are given by, respectively,

$$W^Q_{t_1}(G,t_1) = (T-t_1)\cdot\bar{w} + p_0\cdot\bar{w}^c,$$
$$W^Q_{t_1}(B,t_1) = (T-t_1)\cdot\bar{w} + (1-p_0)\cdot\bar{w}^c.$$

We have $p_0 \geqslant \frac{1}{2}$, hence $W^Q_{t_1}(G,t_1) \geqslant W^Q_{t_1}(B,t_1)$. Using (4), (13), and (14), one can calculate the flow payoff from staying silent until t in an ideal equilibrium, which equals $w\left(\beta_0\left(1-F(t)\right)\right)$. Therefore, the value from not making a report, as evaluated at t_1, equals

$$W^Q_{t_1}(\varnothing) = \sum_{k=1}^{|\mathcal{S}|}(t_{k+1}-t_k)\cdot w\left(\beta_0\left(1-F(t_k)\right)\right) + (T-\bar{t})\cdot w\left(\beta_0\left(1-F(\bar{t})\right)\right) + w^c\left(\beta_0\left(1-F(\bar{t})\right)\right).$$

Staying silent is optimal if and only if $W^Q_{t_1}(G,t_1) \leqslant W^Q_{t_1}(\varnothing)$. Because all terms in $W^Q_{t_1}(\varnothing)$ are finite, this requires that both $\bar{w}$ and $\bar{w}^c$ to be finite. However, if $w(\cdot)$ or $w^c(\cdot)$ is strictly increasing and convex, it must be that $\bar{w} = +\infty$. Then $W^Q_{t_1}(G,t_1) \leqslant W^Q_{t_1}(\varnothing)$ cannot be upheld, so no ideal equilibria exist when either $w(\cdot)$, or $w^c(\cdot)$ is strictly convex. □

Proof of Proposition 8. The proposition follows immediately from the proofs of all referenced statements, which showed that the results continue to hold under the conditions stated in the proposition. □

Proof of Proposition 9. All of the results mentioned in the statement make comparative statements regarding $|i(m,t)|$ for a given m (with the comparison being either across t in most statements, or across equilibria for a given t in Proposition 5). This proof shows that given p_0, there exists an injection $i(m,t) \to i_{KL}(m,t)$ which is strictly increasing for all $i(m,t) > 0$ and strictly decreasing for all $i(m,t) < 0$, which proves the statement of the proposition, since $i(G,t) > 0 > i(B,t)$ for all $t \in \mathcal{S}$ in any equilibrium.

Recall that by Corollary 12, the observer's belief about the state stays constant in the absence of a report: $p_t = p_0$ for all $t \in \mathcal{T}$. This implies that given p_0, both $i(m,t)$ and $i_{KL}(m,t)$ depend solely on $p(m,t)$. Taking the derivative of $i_{KL}(m,t)$ with respect to $p(m,t)$, we have:

$$\frac{di_{KL}(m,t)}{dp(m,t)} = \ln\left(\frac{p(m,t)}{p_{t-1}}\right) - \ln\left(\frac{1-p(m,t)}{1-p_{t-1}}\right) = i(m,t),$$

hence $\frac{di_{KL}(m,t)}{dp(m,t)} > 0 \iff i(m,t) > 0 \iff p(m,t) > p_{t-1}$. In the end, we have that $i(m,t) > 0 \Rightarrow p(m,t) > p_{t-1} \Rightarrow \frac{di_{KL}(m,t)}{dp(m,t)} > 0$, so given that $i(m,t)$ is then a continuous strictly increasing function of $p(m,t)$, we can construct an injection described above. A mirror argument applies for the case $i(m,t) < 0 \iff p(m,t) < p_{t-1}$. $\square$