Only Time Will Tell: Credible Dynamic Signaling*

Egor Starkov†

April 27, 2020

Abstract

This paper explores a model of dynamic signaling without commitment. It is known that separating equilibria do not exist in such settings because actions in any single period are too costless to mimic to be effective as signals, and the sender cannot commit to taking costly actions in the future. This paper, however, shows that informative and payoff-relevant signaling can occur even without commitment and without resorting to unreasonable off-path beliefs. Such signaling can only happen through attrition, when the weakest type mixes between revealing own type and pooling with the stronger types. The possibility of full information revelation in the limit hence depends crucially on the assumptions about the state space. We illustrate the results by exploring a model of dynamic price signaling and show that prices may be informative of product quality, with both high and low prices being able to signal high quality.

Keywords: dynamic signaling, repeated signaling, reputation, attrition

JEL Codes: C73, D82, L15

1 Introduction

In his seminal contribution, Spence [1973] has argued that economic agents’ actions can signal their private information, giving an example of schooling as a signal of ability in labor markets. In the years since, researchers have extensively studied signaling models, describing the fundamental driving forces driving them and identifying signaling patterns in a wide spectrum of applications: from bargaining (Vincent [1990]) and limit pricing (Milgrom and Roberts [1982a,b]) to corporate finance (Leland and Pyle [1977]) and advertising (Milgrom and Roberts [1986])†. However, most signaling models explore static interactions.

Dynamic signaling models may be better suited to explore some applications – for example, the choice of price to signal product quality or the choice of education/effort to signal worker’s ability are both dynamic problems, conceptually. However, signaling in dynamic settings has a salient conceptual problem. In the context of Spence’s story of signaling ability through education, this problem was formulated by Admati and Perry [1987]: “Once a high ability worker has gone to school long enough to distinguish himself from a worker of lower ability, the firms would offer wages

---

*This paper is based on chapter 3 of the author’s Ph.D. thesis. The author thanks Nemanja Antić, Eddie Dekel, Jeffrey Ely, Yingni Guo, Nicolas Inostroza, Johan Lagerlöf, Wojciech Olszewski, Ludvig Sinander, Peter Norman Sørensen, Bruno Strulovici and seminar participants at Northwestern University for valuable feedback and helpful comments.

†Department of Economics, University of Copenhagen, Øster Farimagsgade 5, bygning 26, 1353 København K, Denmark; e-mail: egor@starkov.email.

1See Riley [2001] for an excellent survey of the early literature on signaling.
appropriate to a high ability worker before enough time has elapsed to present an effective screen” (footnote 6). In other words, if neither workers can commit to completing their education in the future, nor firms can commit to avoid hiring undereducated workers, then education cannot serve as an effective signal.\footnote{One possible explanation for the lack of commitment is the possibility of renegotiation; see Beaudry and Poitevin \citeyear{BeaudryPoitevin1993}.}

To give a particular example: when low-ability workers are not supposed to pursue a college degree, a single day spent in that pursuit would imply that a worker’s ability is not low, and market wages for college dropouts would be correspondingly high – too high to actually deter low-ability workers from enrolling in college only to drop out soon thereafter.

The literature has responded to the conceptual challenge described above by searching for aspects of such dynamic interactions which would neutralize the argument above, thus enabling variations of the dynamic signaling model in which static separation is possible. Proposed solutions include: altering the payoffs to add extrinsic motivation for signaling \cite{Weiss1983}, tacit collusion on the receivers’ side to generate instrumental commitment \cite{NoldekeVanDamme1990, Swinkels1999}, mutable types for the sender to allow for stronger reputational threats off path \cite{Roddie2012a, Roddie2012b}, or receivers observing noisy outcomes instead of the sender’s actions \cite{Dilmoe2017, Heinsalu2018}.\footnote{See Whitmeyer \citeyear{Whitmeyer2019} for a discussion regarding the sender-optimal amount of noise in signaling.}

The basic case without any of the above is implicitly perceived as one in which separation is impossible – if an equilibrium even exists, that is. However, the impossibility argument of Admati and Perry \citeyear{AdmatiPerry1987} only applies to perfectly separating outcomes – it does not preclude partial separation, when certain actions act as suggestive rather than conclusive evidence of the sender’s information. The limits of such suggestive signaling in dynamic settings have, to our best knowledge, not been carefully investigated in the literature. We aim to fill this gap.

This paper undertakes the mission to characterize all informative outcomes that can arise in a general model of dynamic (a.k.a. repeated) signaling without commitment. Our main result shows that the scope for signaling, while limited, does in fact exist in dynamic settings, contrary to the intuition of Admati and Perry \citeyear{AdmatiPerry1987}. In particular, payoff-relevant signaling is possible via what is effectively a war of attrition, in which all types of the sender pool on the same action, with the lowest type mixing between pooling and separating to a myopically optimal action. It is therefore possible for higher types to separate from the lowest type as the game continues to infinity, but they cannot separate from each other. Beyond such attrition, information can only be revealed in a way that does not affect the sender’s payoffs: for example, a high and a low type may pool on a strategy which is different from that of some medium type, but only if all three types gain the exact same payoffs from either strategy. This latter kind of information revelation can also be conveyed via cheap talk.

The conclusion regarding the uniqueness of attrition as an informative equilibrium outcome leads into another message of the paper, which is methodological. The mechanism of attrition of the lowest type can yield full separation asymptotically if there are only two types in the game but not if there are more (but finitely many). Further, one can show that in an analog of our model with a continuous type space, full asymptotic revelation is possible again.\footnote{An example of such outcome is presented by Fuchs and Skrzypacz \citeyear{FuchsSkrzypacz2010}.} This aims to show that one’s modelling assumptions may crucially affect the result even when they are about an object that is as seemingly abstract and arbitrary as type space.
While the attrition structure is restrictive, it nonetheless allows for a nontrivial equilibrium multiplicity. In addition to various possible combinations of pooling and separating-through-attrition in different periods, there may be multiple attrition outcomes that can be sustained as equilibria at any given moment. These outcomes would be different in their informativeness, i.e., the probability of separation of the lowest type. This dimension of multiplicity (as well as the range of informative outcomes sustainable in equilibrium in a given period more generally) depends on the richness of the action set.

Our paper formalizes the folk wisdom, in that the attrition equilibrium structure outlined above (“the high type chooses action \( a' \), the low type mixes between \( a' \) and \( a'' \) with positive probabilities”) is encountered in many dynamic applications. However, in spite of the overwhelming presence in the applied theory literature, the issue has never received a rigorous treatment in signaling literature. Our paper amends that, demonstrating that the uniqueness of attrition as an equilibrium structure arises in general settings well beyond the specific models explored previously.

Our analysis relies on the restriction of off-equilibrium path beliefs to be “reasonable”. In particular, we adopt the assumption of non-increasing belief supports or, as labeled by Bond and Zhong [2016], NDOC (“Never Dissuaded Once Convinced”) assumption. As the name suggests, it implies that once the receiver has ruled out some type of the sender as impossible, the receiver stands by this belief and never again assigns positive probability to that type, including off the equilibrium path. This assumption has been criticized as leading to possible equilibrium nonexistence (see Madrigal, Tan, and Werlang [1987] and Nöldeke and van Damme [1990b]). We thus characterize the equilibria conditional on existence, without making any existence claims. Section 5, however, provides some examples of such equilibria.

Kaya [2009] and Roddie [2012a, b] have shown that in the absence of NDOC full instantaneous separation is possible in dynamic settings, since the sender’s behavior can be disciplined by strong reputational threats in case of deviations. While the approach can be justified when the sender’s type may change over time and hence needs constant re-verification, in other settings it is susceptible to a critique of using unreasonable off-path threats to sustain an equilibrium, a practice typically reproved in the literature on equilibrium refinements for static signaling games, as well as equilibrium concepts for dynamic games. Furthermore, it feels tongue-in-cheek to even call such equilibria “separating”, since the sender of some type can never properly separate from other types in that scenario – while the receiver’s belief may assign probability zero to those other types as long as the sender plays by the rules, these types are still perceived as possible, and the sender’s deviation from the equilibrium path would lead the receiver to recognize these types as probable. If any tremble away from the prescribed strategy can ruin all of the sender’s acquired reputation, then what is such reputation worth? In contrast, NDOC assumption allows us to explore the limits of credible signaling – the one which is not reversed by future deviations. Finally, NDOC has been widely used in applied models.


7 Away from Bond and Zhong [2016], one can also find the analogs of NDOC in Grossman and Perry [1986].
To illustrate our results we explore a simple model of price signaling. In this model a firm is privately informed about the quality of its product, and sets the price of the product in every period in an attempt to signal this quality and increase the consumers’ willingness to pay. We construct a family of informative equilibria in this setting and show that both inefficiently low and inefficiently high prices are equally fit to signal high quality in equilibrium, while the literature (some mentioned in Section 5) typically focused on one of the two as a signal. Relatedly, we show that the price path in an informative equilibrium is highly indeterminate (even beyond the dichotomy outlined above), which makes it difficult to test empirically whether price signaling is taking place in a given industry.

The remainder of this paper is organized as follows. Section 2 describes the model. We then proceed to analyze two versions of this model. The two-type version in Section 3 can be seen as an illustrative example. The version with finitely many types is then explored in Section 4. Section 5 considers an application to price signaling and illustrates how our results can be used to construct informative equilibria. Section 6 concludes. All proofs are in the Appendix.

2 Model

2.1 Primitives

We will be looking at a continuous limit of a discrete-time infinite-horizon game. Time is indexed by \( t \in \mathcal{T} \equiv \{0, dt, 2dt, ...\} \); period length \( dt \) is assumed to be arbitrarily small. There is a long-lived agent (sender) who has some persistent type \( \theta \in \Theta \), where \( \Theta \) is a finite ordered set. Alternatively, \( \theta \) can be the state of the world that the agent is informed of.

In every period \( t \) the agent has to choose an action \( a_t \in A \), where \( A \) is some compact set. Agent’s action choices affect public outcome \( x_t \in X \), which is a random process, and its distribution at time \( t \) depends on \( \theta \) and \( a_t \) (more generally, it can depend on the whole past history). We assume that outcomes never allow to perfectly identify \( \theta \): the support of \( x_t \) conditional on \( a_t \) does not depend on \( \theta \). Let \( h_t \equiv \{(a_s, x_s) \mid s \leq t \} \) denote the history of agent’s past action choices and outcomes up to (but not including) time \( t \in \mathcal{T} \). Let \( \mathcal{H}_t \) denote the set of all such time-\( t \) histories and \( \mathcal{H} \equiv \bigcup_{t \in \mathcal{T}} \mathcal{H}_t \) the set of all such histories.

A passive player (receiver), who is also long-lived, starts the game with some prior belief \( p_0 \in \Delta(\Theta) \) about the agent’s type. The receiver updates this belief \( p_t \) every period upon observing the agent’s action choice \( a_t \) and outcome \( x_t \). In shaping this belief the receiver employs Bayes’ rule whenever possible. Off-path beliefs are described further. For specificity, we use \( p(h_t) \) to denote belief after history \( h_t \). Hereinafter, belief \( p_t \) is also referred to as the agent’s reputation. For any \( p_t \), let \( S(p_t) \subseteq \Theta \) be the support of belief \( p_t \), i.e., the set of types to which \( p_t \) assigns positive weight. With abuse of notation, let \( S(h_t) \equiv S(p(h_t)) \).

At the end of every period the agent receives flow payoff \( u^\theta(a_t, p_t) \). For simplicity, we assume that payoff \( u^\theta \) does not depend on outcome \( x_t \) except through reputation \( p_t \). This assumption can

---

8 A similar point was made by Kaya [2013] in relation to signaling product quality through advertising expenditures.

9 One could call \( x_t \) a public “signal”; we avoid this phrasing so as to not create confusion with the process of signaling through actions.
be relaxed as long as the payoff effect of outcomes is weak relative to reputation and the assumption of payoff monotonicity as given in 2.3 is adjusted accordingly. Define a bliss (myopically optimal) action set for type $\theta$ given reputation $p$ as

$$B(p|\theta) \equiv \arg\max_{a \in A} \left\{ u^\theta(a, p) \right\}. $$

All in all, the intra-period timing is as follows: (1) action $a_t$ is chosen; (2) outcome $x_t$ is realized; (3) belief $p_t$ is updated; (4) payoff $u^\theta(a_t, p_t)$ is awarded.

The agent maximizes his expected discounted sum of utilities. A pure strategy for the agent of type $\theta$ is $a^\theta : H \to A$. Given some belief system $p$, let $U^\theta(a^\theta| h_t)$ denote the expected discounted continuation utility of type $\theta$ from following strategy $a^\theta$ starting from $h_t$:

$$U^\theta(a^\theta| h_t) \equiv \mathbb{E}_h \left[ \sum_{s \in T, s \geq t} e^{-r(s-t)}u^\theta(a^\theta(h_s), p(h_s))dt \mid \theta, h_t \right],$$

where $r$ is the agent’s discount rate. The expectation is taken over future histories or, equivalently, over the future outcomes. Strategy $a^\theta$ is optimal for the agent of type $\theta$ given belief system $p(h_t)$ if it maximizes his continuation payoff at every history $h_t \in H$:

$$U^\theta(a^\theta| h_t) = U^\theta(h_t) \equiv \max_a \left\{ U^\theta(a| h_t) \right\}. $$

With a slight abuse of notation we let $U^\theta(h_t \cup a) \equiv \mathbb{E}_x \left[ \max_a \left\{ U^\theta(a| h_t \cup (a, x)) \right\} \right]$, denote the highest expected continuation utility that type $\theta$ can achieve conditional on taking action $a$ at history $h_t$.

Finally, a (behavioral) strategy for the agent of type $\theta$ is $\alpha^\theta : H \to \Delta(A)$. We let $\alpha^\theta(h_t)(a)$ denote the probability with which action $a$ should be played by type $\theta$ after history $h_t$ according to strategy $\alpha^\theta(h_t)$. A pure strategy $a^\theta$ is on path according to some $\alpha^\theta$ if for any $h_t \in H$: $\alpha^\theta(h_t)(a(h_t)) > 0$. A behavioral strategy $\alpha^\theta$ is then optimal for $\theta$ if all pure strategies $a^\theta$ on path according to $\alpha^\theta$ are optimal.

### 2.2 Equilibrium Concept

This is a dynamic game of incomplete information. The solution concept that is least common denominator among concepts used for this kind of games is Perfect Bayesian Equilibrium (PBE). In such an equilibrium, all players maximize their expected continuation payoffs given their beliefs about other players’ actions and beliefs, and these beliefs must be consistent with the players’ knowledge of the game.

**Definition 1.** A Perfect Bayesian Equilibrium is given by the agent’s strategy profile $\alpha = \{ \alpha^\theta \}_{\theta \in \Theta}$ with $\alpha^\theta : H \to \Delta(A)$ and the receiver’s belief system $p : H \to \Delta(\Theta)$ such that:

1. the agent’s strategy profile $\alpha$ is optimal for all types $\theta$;
2. the observer’s belief $p$ is updated using Bayes’ rule whenever possible.

PBE is a maximally permissive solution concept. Our main results characterize signaling in all PBE that satisfy NDOC (as defined in the following subsection), hence they will also apply if one imposes additional restrictions or equilibrium refinements on top of PBE with NDOC.
2.3 Assumptions

The two sections above define the primitives of the model but impose only very minimal restrictions on them. Throughout the paper, we will also impose the following assumptions:

**(MON)** Flow payoff function \(u^\theta(a_t, p_t)\) is weakly increasing in \(p_t\) w.r.t. FOSD mass shifts. I.e., for any \(p', p'' \in \Delta(\Theta)\) such that \(p'(\theta') > p''(\theta')\), \(p'(\theta'') < p''(\theta'')\) for some \(\theta' > \theta''\), and \(p'(\theta) = p''(\theta)\) for all \(\theta \in \Theta \setminus \{\theta', \theta''\}\), it should be that \(u^\theta(a_t, p') \geq u^\theta(a_t, p'')\). Further, for any \(\theta\) and \(a_t\), if \(p_t >_{FOSD} \delta_0\) then \(u^\theta(a_t, p_t) > u^\theta(a_t, \delta_0)\).

**(FIN)** Equilibrium strategy \(\alpha\) has finite support for all \(h_t\).

**(NDOC)** Process \(p_t\) is progressively absolutely continuous. I.e., for any \(h_s \supset h_t\), \(p(h_s)\) is absolutely continuous w.r.t. \(p(h_t)\).

Further, below are some equivalent or related conditions for the above; with the relations between the respective pairs of conditions described in the subsequent text as well as Lemma \(\text{[1]}\).

**(MON-2)** \(\Theta = \{H, L\}\) and \(u^\theta(a_t, p_t)\) is weakly increasing in \(p_t(H)\) and \(u^L(a_t, p_t) > u^L(a_t, \delta_L)\) for all \(a_t\) and \(p_t \neq \delta_L\).

**(FIN-M)** Action set \(A\) is finite.

**(NDOC-P)** After any off-path action \(a\): \(p(h_t \cup (a, x_t)) = \delta_{\min S(h_t)}\) for any \(x_t\).

In the above, \(\delta_\theta\) is the Dirac delta: \(p(h_t) = \delta_\theta\) for some \(\theta \in \Theta\) is equivalent to saying that \(p(\theta|h_t) = 1\) and \(p(\theta'|h_t) = 0\) for all \(\theta' \neq \theta\).

The first assumption, (MON), requires the sender’s flow payoff function to be monotone w.r.t. reputation \(p_t\). It is sufficient to have weak monotonicity (w.r.t. FOSD order on beliefs) with the exception that it must always be strictly beneficial to pool with the higher types. This assumption will be imposed throughout, and in the two-type model (\(|\Theta| = 2\)) it will also be the only restriction on payoffs needed for the result. Further, in case of two types it can be written more simply as (MON-2). In other words, if \(|\Theta| = 2\) then (MON) and (MON-2) are equivalent.

The next pair of assumptions is (FIN) and (FIN-M). The former is an equilibrium refinement that demands that the sender’s equilibrium strategy has finite support at any history. The latter is a restriction on the model that in the action set of the sender is finite. The analysis in the remainder of the paper relies on one of these two conditions to hold in order to avoid the problem of Bayesian inference from zero probability events. This problem is illustrated by the following example.

**Example 1.** Let \(\Theta = \{L, H\}\), \(A = [0, 1]\), and suppose that outcomes are uninformative: \(x_t \equiv 0\). Fix some equilibrium and history \(h_t\) therein. Suppose that the low type’s strategy \(\alpha^L(h_t)\) assigns weight one to action \(a = 0\), while \(\alpha^L(h_t)\) mixes uniformly over all actions \(a \in [0, 1]\). By Bayes’ rule, the receiver’s belief at history \(h_{t+dt} = h_t \cup (0, 0)\) must assign probability one to type \(L\) and probability zero to type \(H\). However, it is not immediate whether the receiver must in such cases rule out type \(H\) at all histories following \(h_{t+dt}\) (which he does if (NDOC) holds).

It is immediate that (FIN-M) is sufficient for (FIN) to hold, therefore in the remainder of this paper we use (FIN). However, (FIN-M) is a useful reminder that in finite games no further equilibrium refinements are required, apart from (NDOC).

\(^{10}\)The strict part of (MON) simplifies the analysis but rules out some relevant cases. E.g., it disallows the payoff to be a step function, which is the case when the agent only cares about his reputation being above some cutoff.
Finally, (NDOC) is the assumption on equilibrium beliefs that drives our analysis. In particular, it says that if \( p(\theta|h_t) = 0 \) then \( p(\theta|h_s) = 0 \) for any \( h_s \supset h_t \). Note that this applies both on and off the equilibrium path. For the discussion of (NDOC), refer to Secton 1. To simplify the analysis, we strengthen (NDOC) to (NDOC-P), which requires that off the equilibrium path, the receiver’s belief \( p(h_t) \) must be pessimistic – it must put all weight on the lowest type among those not yet ruled out by the receiver. Given (MON), this condition imposes the strongest possible punishment on the sender for any deviation, among those punishments that satisfy (NDOC). Therefore, we argue that for any equilibrium that satisfies (NDOC), there exists an equivalent one that satisfies (NDOC-P), despite the latter being a stronger condition.

The claims made in this section are summarized in the following lemma.

**Lemma 1.** Model assumptions are connected through the following relations.

1. If \( |\Theta| = 2 \) then (MON) and (MON-2) are equivalent.
2. (FIN-M) implies (FIN).
3. If (MON) holds then for any equilibrium that satisfies (FIN) and (NDOC), there exists a payoff-equivalent and on-path strategy-equivalent equilibrium that satisfies (FIN) and (NDOC-P).

### 2.4 Discussion of Assumptions

This section discusses the assumptions that are implicit in the model set-up so that the reader can get a clearer picture of which aspects of the model are important for the results, and which modelling assumptions were made purely for expositional simplicity.

To start with, the model setup includes a number of assumptions that impede with instantaneous separation of types, namely: persistent sender’s type \( \theta \), vanishing period length, compact action set and finite action costs. The former is required for (NDOC) to have any bite: if type could change over time then the sender’s reputation would need constant re-verification, meaning that credible signaling is impossible by design. The other three assumptions are meant to remove any implicit commitment power the sender may have (since in discrete time he can effectively commit to not revise his action until the next period) and to remove the potential of any given single action to be informative. All of these assumptions restrict us to the world in which, according to Weiss [1983] and Admati and Perry [1987], perfect separation is impossible, since this is the world we are interested in exploring.

This paper’s message is not about arguing that this is the only plausible set of assumptions in dynamic signaling. Indeed, there are many settings in which a single action has the weight to be informative enough by itself, or the sender has at least some commitment power. This paper argues instead that there exist real-world settings to which the above set of assumptions applies, and we as economists care about characterizing them. For example, one such setting is price signaling by the firm – be it signaling of the firm’s product quality to consumers or signaling of its production costs to existing and potential competitors. Prices are typically perfectly observable and can be changed frequently at no cost.

The above raises the question of why we limit ourselves to discrete time (thus giving the sender limited commitment power) rather than exploring a proper continuous time model. The answer is simplicity. While the essence of our results carries over to the continuous time case, their statements...
become less clear-cut, and the analysis of such model becomes encumbered by the specifics of the continuous-time analysis.

On the other hand, finiteness of the type space is a crucial assumption. In particular, if type \( \theta \) is distributed on an interval then attrition takes a very different form from what is stated in Theorems 1 and 2. Instead of the lowest type separating with positive probability in every period (in which payoff-relevant signaling happens), we could have a positive mass of types at the lower end of the support separating every period. In continuous time, the lowest bound of the support of types would increase smoothly over time along the pooling path, shrinking the support; an example of such equilibrium is constructed by Fuchs and Skrzypacz [2010]. Importantly, such attrition could lead to full separation in the limit as \( t \to \infty \), unlike in the case with finitely many (but more than two) types. Furthermore, we can no longer guarantee that with a continuum of types, attrition is the only way in which payoff-relevant signaling can proceed.

Finally, we assume that the receiver is passive and the sender receives utility from reputation \( p \). One may see this as a reduced form of a repeated Stackelberg game in which in every period \( t \) the sender first chooses action \( a_t \), which together with the respective outcome \( x_t \) determines his current reputation \( p_t \), and the receiver then responds with some action \( b_t \), after which both players \( i \in \{ S, R \} \) receive utilities \( u_i^\theta (a_t, b_t) \). This is a standard reduction used both in signaling literature (Kaya [2009], Roddie [2012a,b]) and other literatures (e.g., Bayesian Persuasion – see Kamenica and Gentzkow [2011]). Given that this is by now a standard technique, we do not describe the full game in order to economize on notation. However, our results can be easily extended to both repeated Stackelberg games in which the sender and the receiver act in sequence in every period, and (with slightly more effort) to repeated games in which both act simultaneously.

3 Two Types

This section explores the version of the model with only two types: \( \Theta = \{ L, H \} \). Here we show that signaling must take the form of attrition regardless of payoffs, as long as they are monotone in reputation \( p_t \). The first part of Theorem 1 states that perfect separation cannot occur at any history in equilibrium: if a given action is on path for \( \theta = H \) then it is also on path for \( \theta = L \). This statement captures the idea of Admati and Perry [1987] and Nöldeke and van Damme [1990a]. We also observe that such pooling action must be effectively unique, in the sense of all pooling actions being payoff-equivalent for all types of the agent – this follows trivially from the fact that both types must be indifferent between playing any such action if there are more than one.

The new insight is that the converse to the first statement is not necessarily true: if \( \alpha^L (a|h_t) > 0 \) then \( \alpha^H (a|h_t) \) may or may not be positive. In other words, there may exist actions which perfectly identify the low type, even if there do not exist any that identify the high type. It is immediate that the low type must be mixing for this to be possible. All this is summarized by the second part of the theorem. The statement does not claim existence of any such separating actions, since they, of course, need not exist in any given case. However, Section 5 presents an example of a setting in which such informative equilibrium exists.

---

11 The receiver’s utility may depend on true \( \theta \) as long as the receiver does not observe his own utility flow.
Theorem 1. Suppose that $|\Theta| = 2$, dt is small enough, and (MON-2) holds. In any equilibrium such that (FIN) and (NDOC) hold, at any $h_t \in H$ with $S(h_t) = \{L, H\}$, and for any $a \in A$:

1. If $\alpha^H(a|h_t) > 0$ then $\alpha^L(a|h_t) > 0$. Further, all such $a$ are payoff-equivalent in the sense that $U^\theta(h_t \cup a)$ is the same across such $a$ for all $\theta$.

2. If $\alpha^H(a|h_t) = 0$ and $\alpha^L(a|h_t) > 0$ then $a \in B(\delta_L|L)$ and $U^L(h_t \cup a) = U^L(h_t \cup a')$ for any $a'$ such that $\alpha^H(a'|h_t) > 0$.

Note that the attrition structure of signaling imposes strong restrictions on actions that can be played in equilibrium. Firstly, any separating action perfectly identifies the low type, meaning it cannot bear any signaling incentives, and must hence be myopically optimal for the low type. Secondly, if the low type mixes between pooling and separating, then he must be indifferent between the two – meaning that pooling with the high type must yield exactly the same expected payoff for the low type as separation. Gains from pooling in this scenario (higher reputation) are exactly offset by the cost of taking suboptimal actions in current and/or future periods.

It is worth emphasizing that the result holds under very minimal assumptions on payoffs and signals: the only requirements imposed on the model are that the sender’s payoff is increasing in $p$ (which, in fact, is only required for the low type) and that the outcomes $x$ are not perfectly revealing. In other words, if your game fits the following framework:

- dynamic game with continuous time or short time intervals or patient players,
- binary state of the world known by one player but not other(s),
- the informed player has a strict preference over other(s)’ belief, and the direction of this preference does not change with the state,
- the informed player chooses an action every period but cannot verifiably reveal the state,

then the only informative equilibrium structure that can arise in this game (unless you are willing to allow for NDOC-nonconformant beliefs off the equilibrium path) is attrition. Under attrition, the high type is playing some pooling action, while the low type mixes between that and a separating action.

4 Finite Types

We now move to exploring the setting with more than two but finitely many types. In this section we show that the insight of Theorem 1 can be extended to this case, although allowing for many types does raise a number of additional issues and calls for additional assumptions.

4.1 Single-Crossing

In order to secure the result in case of many types, we need to impose the following extra assumption on payoffs:

(SC) $U^\theta(a|h_t)$ satisfies single-crossing in $(\theta, a)$ at all $h_t$, i.e., for any $a', a'' \in \cup_{h_t} \cup_{\theta \in S(h_t)} \arg \max_a U^\theta(a|h_t)$ and all $h_t$, function $U(\theta) \equiv U^\theta(a''|h_t) - U^\theta(a'|h_t)$ either crosses zero at most once, or is identically zero.

This assumption belongs to a family of single-crossing conditions widely encountered in the
literature on signaling, monotone comparative statics, and mechanism design.\footnote{Cf. Laffont and Martimort \[2002\], p.35. Classic references on MCS include Milgrom and Shannon \[1994\] and Athey \[2002\].} The purpose of our condition is standard: to ensure that the agent’s preferences over strategies satisfy a kind of monotonicity w.r.t. his type. Our condition, however, has three distinctive features which differentiate it slightly from other single-crossing conditions in the literature.

\textit{Feature 1.} This is an assumption about an equilibrium object, since belief system $\rho(h_t) – which enters $U^\theta(a|h_t)$ – is endogenous to equilibrium. The simplest way to justify the assumption in this respect is strengthening the assumption to all combinations of actions and beliefs $(a, \rho)$, i.e., to assume that

$$E_h \left[ \sum_{s \in T, s \geq t} e^{-r(s-t)} u^\theta(a(h_s), \rho(h_s)) dt \mid \theta, h_t \right],$$

is single-crossing for all pairs $(a', \rho')$, $(a'', \rho'')$.

\textit{Feature 2.} Unlike most standard single-crossing assumptions, (SC) does not require that $a'' > a'$ (and that single-crossing happens from below)\footnote{The order implied here is the product order on the set $(A \times \Delta(\Theta))^H$ of all collections $(a(h_t), \rho(h_t))$, composed of some order on $A$ (although we have not explicitly assumed that $A$ is ordered) and FOSD order on $\Delta(\Theta)$.} In contrast, we also require the statement to hold for any pair of unordered strategies and respective belief profiles. The reason for that is while the grand set of choices $(A \times \Delta(\Theta))^H$ is a lattice w.r.t. the product order (for some arbitrary order on $A$), the subset of these choices available to the sender at any given history is not necessarily its sublattice. The standard monotone comparative statics results are then of limited use. On the upside, however, we only require that single-crossing is satisfied for those strategies that may be optimal for any type. I.e., if one can rule out some actions as certainly suboptimal at certain histories, these actions may be safely ignored.

\textit{Feature 3.} (SC) is a condition on the discounted sum $\sum_t e^{-rt} u^\theta(a_t, \rho_t)$ rather than on the flow utility $u^\theta(a, \rho)$. While the latter would be more preferable, aggregating single-crossing is not a trivial problem. Quah and Strulovici \[2012\] discuss this problem and offer possible solutions, but none of them apply to our setting due to feature 2 above.

All of the above means that (SC) is quite a non-trivial condition and may be difficult to verify in many models. If anything, we foresee that verifying (SC) would be the main impediment to applying our result in the models. However, this task is not impossible. In Section 5 we provide an example of an applied model with a payoff function that is easily shown to satisfy (SC).

4.2 Attrition Structure of Equilibrium Signaling

Theorem 2 that we gradually build up to is the analog of Theorem 1 for the case when $|\Theta| > 2$, in the sense of characterizing the actions available in equilibrium at any history. We begin, however, by stating a weaker result which provides a clearer characterization of the attrition structure of equilibrium signaling with $|\Theta| > 2$. Proposition 1 below establishes that as long as (SC) and other previously stated assumptions hold, there are effectively two classes of strategies played in the game. The first class consists of the pooling strategies played by all types. While there may be many such strategies, they must all be payoff-equivalent, so this class is, in a sense, degenerate. The second class is that of separating strategies employed by the lowest type – these may differ in
which pooling strategies they mimic and for how long. However, all of them are only ever played by the currently-lowest type.

To state this and following results we need to introduce some additional notation and definitions. Firstly, denote the two boundaries of the belief support as \( S(h_t) \equiv \max S(h_t) \) and \( S(h_t) \equiv \min S(h_t) \) respectively. Furthermore, in a manner similar to type support \( S \), given an equilibrium strategy profile let us define action support as

\[
A^\theta(h_t) \equiv \left\{ a \in A | \alpha^\theta(h_t)(a) > 0 \right\},
\]

\[
A(h_t) \equiv \cup_{\theta \in S(h_t)} A^\theta(h_t).
\]

We say that a pure strategy \( a \) arrives at \( h_t = \{a_s(h_t), x_s(h_t)\}_{s \in \tau, s \leq t} \) and denote it as \( a \supset h_t \) if \( a(h_t') = a(h_t) \) for any \( h_t' = \{a_s(h_t), x_s(h_t)\}_{s \in \tau, s \leq t'} \) with \( t' < t \). Further, say that \( a \) is on path for \( \theta \) at \( h_t \) if \( a \supset h_t \) and \( a \) is on path according to type \( \theta \)'s equilibrium strategy \( \alpha \) starting from \( h_t \), and \( a \) is on path at \( h_t \) if it is on path at \( h_t \) for some \( \theta \in S(h_t) \). Note that action supports and “on-pathness” only work properly if (FIN) is satisfied. For example, if some \( \alpha^\theta(h_t) \) prescribed mixing uniformly over an interval of actions \( a \in [0,1] \) in violation of (FIN), then none of these actions would be included in the support (since they are all played with zero probability) and none of them would be on path, according to the definitions above.

We proceed by defining payoff equivalence of strategies in a straightforward manner.

**Definition 2.** Fix an equilibrium and history \( h_t \). Any two pure strategies \( a', a'' \supset h_t \) are:

- payoff-distinct at \( h_t \) if there exists \( \theta \in S(h_t) \) such that \( U^\theta(a'|h_t) \neq U^\theta(a''|h_t) \);
- payoff-equivalent at \( h_t \) if they are not payoff-distinct at \( h_t \).

Note also that while using the notation for full pure strategies, throughout the whole analysis we actually work with continuation strategies from some history \( h_t \). When discussing strategies conditional on some history \( h_t \) we ignore all game paths that are ruled out by \( h_t \). In particular, two pure strategies \( a', a'' \supset h_t \) that prescribe the same actions at all \( h_s \supset h_t \) but differ at some \( h_s \supset h_t \) are treated as the same strategy for all means and purposes. We avoid introducing the continuation strategies explicitly in order to economize on notation, which is quite heavy as is.

The result can now be stated as follows.

**Proposition 1.** Suppose the payoff function \( u^\theta \) satisfies (MON) and (SC). Fix an equilibrium such that (FIN) and (NDOC) hold. Fix some history \( h_t \). Then, defining \( \bar{\theta} \equiv S(h_t) \), the following hold:

1. all pure strategies \( a' \) on path at \( h_t \) for any \( \theta \in S(h_t) \backslash \bar{\theta} \) are payoff-equivalent and optimal for all \( \theta \in S(h_t) \) at \( h_t \), and at least one of these strategies is on path for \( \bar{\theta} \) at \( h_t \);
2. any pure strategy \( a'' \) that is on path at \( h_t \) and payoff-distinct at \( h_t \) from any such \( a' \) is only on path for \( \bar{\theta} \).

The proposition implies, in particular, that any pure strategy \( a \) that is on path for some type \( \theta \in S(h_t) \) is also on path for the currently-lowest type \( \bar{\theta} \). Therefore, no type of the agent can ever conclusively separate from \( \bar{\theta} \). At the same time, there may exist strategies that separate \( \bar{\theta} \) away from the remaining types. The weight that the receiver’s belief assigns to \( \bar{\theta} \) may thus decrease over
time along the pooling path of play – it may even converge to zero asymptotically as $t \to \infty$, but it may never become exactly zero.

The proposition above is stated in terms of strategies rather than actions, and so provides only limited insight into how equilibrium actions look in any given period. That is, if $a'$ as defined in the proposition is unique then it is relatively straightforward that in every period there will be some single pooling action as prescribed by $a'$ that all types above $\theta$ will play for sure, while the lowest type will somehow mix between this pooling action and some number (between zero and infinity) of separating actions. The challenge, however, comes from possible non-uniqueness of $a'$. If there are many pooling actions then they may be informative even though (or exactly because) all types are indifferent between them. The following sections explore this issue in more detail and works around it to characterize the within-period signaling outcomes.

4.3 Payoff-Relevant Signaling

To talk about signaling in relation to individual actions (rather than whole strategies), we need to define what exactly “signaling” means in a dynamic context with many types. It is clear that if all types pool on the same action in a given period then no information is revealed, while if every type plays an action different from all others then the full separation occurs, which is the most informative signaling outcome. The two grey zones are partitioning – when, for example, some types play action $a'$ and some others play $a''$ – and mixing, – when one type plays action $a'$ for sure and another type mixes between two actions $a'$ and $a''$. Both of the aforementioned outcomes are usually dubbed as “semi-separation” in static settings and considered informative outcomes in signaling models. In dynamics, however, there are further complications.

In a dynamic setting, it matters not only what action the sender plays in a given period, but also his past and future actions, which can negate the payoff consequences of today’s action. This is illustrated by the following example.

Example 2. Suppose $\Theta = \{1, 2, 3\}$, types are ex ante equiprobable, $A = \mathbb{R}_+$, and $u^\theta(a, p) = \mathbb{E}_p(\theta) - a$. Then the following would be an equilibrium: type $\theta = 2$ plays some $a''$ at $t = 0$ and $a = 0$ in all $t \geq dt$, while types $\theta = 1, 3$ play $a' = a''(1 - e^{-rdt})$ at all $t \geq 0$. This is a PBE of the game as long as $a'' \leq 1$. In this PBE some information about type is conveyed in period zero – namely, type $\theta = 2$ separates from $\theta = 1, 3$, but this signaling is not relevant to the sender’s payoff.

However, the same is not necessarily true for the receiver. In particular, we can think of this example as a game between a worker (sender) and a firm (receiver), where $a$ is the worker’s effort and $\theta$ is his ability. Suppose the receiver’s flow payoff is given by $v(a, p, \theta) = \theta a^2 - \mathbb{E}_p(\theta)$, with the first term being the worker’s output, and $\mathbb{E}_p(\theta)$ in the payoffs of both players is the worker’s wage, dictated by the market. In this case the firm’s expected discounted profit from hiring a worker of type $\theta = 2$ equals $2(a'')^2 - \frac{2}{1-e^{-r\theta}}$, while that from hiring a worker of type $\theta \in \{1, 3\}$ is $\frac{2(a''(1-e^{-r\theta}))^2 - 2}{1-e^{-r\theta}} = 2(a'')^2(1 - e^{-r\theta}) - \frac{2}{1-e^{-r\theta}}$, which is strictly less.

The example above illustrates that arbitrary information can in principle be conveyed via payoff-irrelevant signaling – when different types play different actions, but nonetheless all types are indifferent between all actions and respective continuations. This can be seen as a kind of “cheap talk”: for small $dt$, actions in a single period are effectively costless, and so can be used
as a pure communication device with no regard for action costs. Situations in which informative communication arises through cheap talk have been studied extensively, see Sobel [2013] for a recent survey. Further, one can claim that there will often be scope for such cheap/payoff-irrelevant communication in our model. This is due to multi-dimensionality of reputation \( p \): with more than two types there are bound to be situations in which an agent is indifferent between two kinds of average reputation – one that says the agent is of average type for sure and the one that says the agent is of either high or low type with comparable probabilities. Consequently, in this paper we focus on payoff-relevant signaling – communication that relies on heterogeneity across the agent’s types of costs or benefits from different actions, in the spirit of Spence [1973].

Our contribution to characterizing the informative outcomes in dynamic signaling games without commitment should then be seen as complementary to that of the cheap talk literature.

We now provide the formal definitions of payoff-relevant and irrelevant signaling in our setting.

Definition 3. Fix an equilibrium and history \( h_t \).
- Payoff-relevant signaling happens at \( h_t \) if there exist \( a', a'' \in A(h_t) \) and \( \theta \in S(h_t) \) such that \( U^\theta(h_t \cup a') \neq U^\theta(h_t \cup a'') \).
- Payoff-irrelevant signaling happens at \( h_t \) if there exist \( a', a'' \in A(h_t) \) such that \( p(h_t \cup a') \neq p(h_t \cup a'') \) but \( U^\theta(h_t \cup a') = U^\theta(h_t \cup a'') \) for all \( \theta \in S(h_t) \).

In other words, payoff-relevant signaling implies that at a given history \( h_t \) there are two distinct actions on path, \( a' \) and \( a'' \), and there is some type of the agent for which the choice between these two actions has payoff consequences. Note that since both actions are on path, it cannot be the case that all types prefer one over another – both \( a' \) and \( a'' \) must be optimal for some types of the agent. Payoff-relevance of this action choice is then defined as some type \( \theta \in S(h_t) \) having strict preference between the two, while if all types are indifferent, we say that any signaling that may occur via the choice of \( a' \) over \( a'' \) or vice versa is payoff-irrelevant.

4.4 From Action Profiles to Actions

We are now ready to state the theorem that characterizes payoff-relevant signaling in terms of actions, making the implications of Proposition 1 more explicit. The result below expands the message obtained in Theorem 1 to the case of finitely many types, albeit at the cost of restricting model scope to payoff functions that satisfy (SC).

Theorem 2. Suppose the payoff function \( u^\theta \) satisfies (MON) and (SC). Fix an equilibrium such that (FIN) and (NDOC) hold. Fix some history \( h_t \). If payoff-relevant signaling happens at \( h_t \) then, defining \( \Theta \equiv S(h_t) \), the following hold:

1. any on-path action \( a \in A(h_t) \) is on path for \( \Theta \) at \( h_t \);
2. \( A(h_t) \cap B(\delta(h_t) \Theta) \) is nonempty, and any \( a \) in the intersection is on path only for \( \Theta \) at \( h_t \);
3. any action \( a \in A(h_t) \backslash B(\delta(h_t) \Theta) \) is optimal at \( h_t \) for all \( \Theta \in S(h_t) \).

What the theorem says is that in any equilibrium with payoff-relevant signaling, there are effectively at most two types of actions (as opposed to strategies in Proposition 1) on path at any

\[ \text{[Battaglini] 2002 and [Chakraborty and Harbaugh] 2007, 2010] explore cheap talk with multidimensional information (which our setting is an instance of in case |\Theta| > 2) and show that there generally exist informative equilibria in which the receiver perfectly learns about } N - 1 \text{ out of } N \text{ dimensions of uncertainty.} \]
history – pooling actions (typical element \( \bar{a} \)) and separating actions (typical element \( a \)). The latter are only ever played by the currently-lowest type \( \theta = S(h_t) \) and separate him from the remaining types. As in Theorem 1, any separating action must be myopically optimal for the lowest type given that he is revealed.

Pooling actions, on the other hand, are optimal for all types. Further, if no payoff-irrelevant signaling takes place then any pooling action is, in fact, on path for all \( \theta \in S(h_t) \) – i.e., all types do actually pool on the pooling action(s). There is, however, a subtlety, in that both payoff-relevant and payoff-irrelevant signaling may occur simultaneously at a given history. In that case there will be more than one pooling action, and while all of them are necessarily on path for \( \theta \), higher types may select different actions despite all types being indifferent between all of these actions (and the continuations they induce). Despite payoff-irrelevance for the sender, such separation is still informative for the receiver in that his belief \( p_t \) splits in a non-trivial way across these pooling actions, and therefore such signaling may in principle be payoff-relevant for the receiver.

The corollary below relates to the situations when payoff-relevant signaling occurs at neighboring histories. It states that the pooling action (in the earlier history) in this case must be such that the low type is indifferent between separating and pooling – meaning that the flow payoffs the low type gets from the two actions must be the same. The idea is simple: the low type must be indifferent between separating at \( t \) and \( t + dt \) – meaning that one period of pooling must be exactly as attractive as one period of being identified as \( \theta \). In practice, this means that pooling action must be costlier for \( \theta \) than the separating action, since the former yields higher reputation payoff.

**Corollary 1.** Suppose the conditions in Theorem 2 hold. Suppose payoff-relevant signaling occurs also at \( h_{t+dt} \equiv h_t \cup (\bar{a}, x) \) for some \( \bar{a} \) and all \( x \) in the support. Then such \( \bar{a} \) must satisfy

\[
\mathbb{E}_x \left[ u^\theta(\bar{a}, p(h_t \cup (\bar{a}, x))) | \theta \right] = u^\theta(\bar{a}, \delta_{\bar{a}}).
\]

Finally, the theorem applies to all histories, including those off the equilibrium path. Applying it inductively starting from the root history, we get the following corollary, which states that in the absence of payoff-irrelevant signaling, only the lowest type can ever separate from the rest, while the remaining ones can never separate from one another. It is worth noting that there may be histories \( h_t \) at which \( p(h_t) \) assigns arbitrarily small weight to the lowest type. What is important is that this type can never be ruled out completely along the pooling path.

**Corollary 2.** In any equilibrium in which no payoff-irrelevant signaling happens, for any on-path history \( h_t \), one of the following must hold:

1. \( S(h_t) = \Theta \);
2. \( S(h_t) = \{ \min \Theta \} \).

The corollary above together with Theorem 2 effectively provide a cookbook on how to construct an equilibrium with payoff-relevant signaling only. Suppose we want signaling to occur during the time interval \([0, T]\). Then in every period along the pooling path we shall have two actions available to the sender: a separating action \( a \in B(\delta_{\bar{a}}|\theta) \) only taken by the lowest type \( L \equiv \min \Theta \) and a pooling action \( \bar{a} \) that satisfies the condition in the last part of the theorem – the latter action will be played by \( L \) with some probability and by all other types for sure. Note that we have a degree of freedom in this construction: reputation from taking a pooling action depends on the probability with which type \( L \) separates in that given period. Hence by changing these probabilities we will be
able to sustain different pooling actions $\bar{a}$ in equilibrium. Finally, we need to verify that from time $T$ onwards, the pooling strategy is such that $L$ is exactly indifferent at $T$ (or the last period before $T$) between separating and following this pooling path.

The following section takes this cookbook and uses it to construct an informative equilibrium in a concrete setting.

5 Application to Price Signaling

5.1 Background

This section looks at a simple model of price signaling with product reviews. Price signaling is a phenomenon that is widespread in the real world – high-quality products may be priced at a premium to signal quality, or, conversely, they may offer more free trials or giveaways to help consumers learn about the product. Models exist that support both kinds of behavior. For example, Bagwell and Riordan [1991] show that if some consumers are initially informed of product quality while others learn from repeated purchases, then high and declining prices signal product quality. They also refer to empirical cases which support their conclusions. On the other hand, Vettas [1997] demonstrates that in the presence of social learning, high-type firm prices low on entry, gradually increasing the price afterwards, which is another pattern commonly observed in reality.

While this apparent contradiction – that both high and low introductory prices can serve to signal quality – has been recognized in the literature (see, e.g., Kirmani and Rao [2000]), we are not aware of a theory that addresses it. The simple dynamic model below amends this and shows that both high and low prices are equally fit to serve as (suggestive yet inconclusive) signals of high quality in an informative equilibrium.

5.2 Model Setup

There is a long-lived firm $i$ that faces a continuum of consumers $j \in [0,1]$ every period $t \in \{0, dt, 2dt, \ldots\}$, where period length $dt$ is “small”. The firm offers for sale a single product of privately known quality $\theta \in \Theta = \{H, L\}$ (in Section 5.4 we discuss the case when $|\Theta| > 2$). The marginal costs of production are zero. In every period the firm sets the price $a_t$ of its product and the consumers decide whether to purchase it or not. A consumer’s payoff from buying the product is given by $\theta v_j - a_t$, where $v_j \sim i.i.d. U[0,1]$ is the consumer’s value for quality. Payoff from not buying the product is zero. Consumer $j$ then buys the item if and only if $E[\theta] \geq a_t v_j$.

The population of consumers is renewed every period. The newly arriving consumers base their belief $p(h_t) \equiv p(\theta | h_t)$ about the product quality on the prior $p_0$, the whole price path $\{a_s\}_{s \leq t}$, and product reviews as described below. With probability $1 - e^{-\phi dt} \approx \phi dt$ for $\phi \in [0, 1)$ the population of consumers in a given period generates an informative review $x_t = \theta$ which perfectly reveals the firm’s quality and is observable by all future consumers. With complementary probability $e^{-\phi dt}$ no review is generated: $x_t = \emptyset$.

16The analysis carries over fully to the case when the review arrival rate depends on the number of consumers who purchased the product in a given period.
The flow profit of a firm of type $\theta$ receives in period $t$ after setting price $a_t$ is then given by

$$u^\theta(a_t, p_t) \equiv a_t \left(1 - \frac{a_t}{\mathbb{E}[\theta|p_t]}\right).$$  \hfill (2)

According to Theorem 1 (payoff-relevant) signaling must necessarily take the form of attrition. Under such attrition signaling, type $\theta = L$ mixes between some pooling price $a_t^p$ and separating to bliss price $a_t^{L|L} = \frac{L}{4}$, whereas type $H$ sets the pooling price. We then show that Theorem 2 applies as well, meaning that the result translates to the case $|\Theta| > 2$.

### 5.3 Equilibrium

We will be looking for an equilibrium of the game that satisfies (NDOC-P) and (FIN). In particular, let us construct an equilibrium with informative prices at every history in which the firm’s type is not perfectly known (i.e., unless $p(h_t) \in \{0, 1\}$). If the firm is believed to be bad ($p(h_t) = 0$) then there are no signaling motives – it sets price $a_t^{\theta|L} \equiv \frac{L}{2}$ and earns profit $u^\theta(a_t^{\theta|L}, \delta_L) = \frac{L}{4}$. Similarly, if $p(h_t) = 1$ then $a_t^{\theta|L} \equiv \frac{H}{2}$ and $u^\theta(a_t^{\theta|H}, \delta_H) = \frac{H}{4}$. The latter is due to (NDOC) – without this assumption we could enforce a wide spectrum of prices using the threat of losing reputation in case the firm deviates from some prescribed price level. With (NDOC) such threats are ruled out, hence the firm’s behavior at histories with $p(h_t) = 1$ must be myopically optimal.

We begin by deriving the pooling price $a_t^p$ that renders the low type indifferent between separating and pooling. The former yields the continuation value equal to

$$U^L(a_t^{L|L}|h_t) = \frac{dt}{1-e^{-r dt}} \frac{L}{4} \approx \frac{L}{4r},$$

since $1 - e^{-r dt} \approx r dt$ is a valid approximation when $dt$ is small enough. Pooling, in turn, yields

$$U^L(a_t^p|h_t) = a_t^p \left(1 - \frac{a_t^p}{\mathbb{E}[\theta|p_t]}\right) dt + e^{-r dt}\mathbb{E}_x U^L(h_t \cup (a_t^p, x_t)), \hfill (3)$$

where $p_t \equiv p(h_t \cup (a_t^p, \emptyset))$ and the continuation value can be written as

$$\mathbb{E}_x U^L(h_t \cup (a_t^p, x_t)) = (1 - e^{-\phi dt})U^L(h_t \cup (a_t^p, L)) + e^{-\phi dt}U^L(h_t \cup (a_t^p, \emptyset)).$$

After a bad review $x_t = L$ the consumers are sure that the firm is bad, i.e., $p(h_t \cup (a_t^p, L)) = 0$, so we have $U^L(h_t \cup (a_t^p, L)) = \frac{L}{4}$. After no review $x_t = \emptyset$ the consumers’ belief is inconclusive, hence in our construction signaling should continue. This means $L$ must be indifferent between pooling and separating once again, so $U^L(h_t \cup (a_t^p, \emptyset)) = \frac{L}{4}$ as well. Finally, $L$’s indifference at $h_t$ yields $U^L(a_t^p|h_t) = U^L(a_t^{L|L}|h_t) = \frac{L}{4}$. Plugging all of these into (3), we obtain that flow payoff from pooling must coincide with that from separating:

$$\frac{L}{4} = a_t^p \left(1 - \frac{a_t^p}{\mathbb{E}[\theta|p_t]}\right).$$
The solution to the above is given by

\[ a_t^p = \frac{\mathbb{E}[\theta|p_t]}{2} \left[ 1 \pm \sqrt{1 - \frac{L}{\mathbb{E}[\theta|p_t]}} \right]. \tag{4} \]

Therefore, for a fixed \( \mathbb{E}[\theta|p_t] \) we have two equivalent candidates for the pooling price \( a_t^p \). The negative root corresponds to signaling by setting a low price – below \( L \)’s preferred price (for any reputation). Such pooling price can be seen as low entry pricing à la [1997]. The positive root, conversely, corresponds to the price well above the optimum, and signaling through exclusivity in the spirit of [Bagwell and Riordan 1991].

Further, the pooling output \( a_t^p \) (whichever root to (4) we choose) is a function of the seller’s reputation

\[ p(h_t \cup (a_t^p, \emptyset)) = \frac{p(h_t)}{1 - \lambda_t(1 - p(h_t))}, \tag{5} \]

where \( \lambda_t \) is the probability with which type \( L \) separates at \( h_t \). In particular, we can choose these probabilities freely and construct an equilibrium for arbitrary \( \lambda_t \). Conversely, if we are restricted in our choice of \( a_t^p \) – e.g., if this pooling price must amount to an integer number of dollars, – this constrains the set of \( p(h_t \cup a_t^p) \) and, consequently, \( \lambda_t \) that we can implement in equilibrium at any given history \( h_t \).

Therefore, in general the pooling price path is indeterminate given the market conditions (seller’s reputation), so inferring whether price signaling is taking place in a given market by looking at price data is a daunting task. This point was originally raised by Kaya [2013] in relation to advertising expenditures.

To complete the equilibrium description we only need to argue that setting the pooling price \( a_t^p \) is optimal for \( H \). His continuation value from doing so is

\[ U^H(a_t^p|h_t) = \frac{L}{4} dt + e^{-r dt} \left[ (1 - e^{-\phi dt}) \frac{H}{4r} + e^{-\phi dt} U^H(h_t \cup (a_t^p, \emptyset)) \right]. \]

Indeed, his flow payoff is the same as for \( L \) (the two types only differ in the reviews they get), and in case a good review is generated at \( h_t \), he will be receiving \( \frac{H}{4} \) in every future period. The same, however, applies to any history with \( p(h_t) \in (0, 1) \), hence \( U^H(h_t \cup (a_t^p, \emptyset)) = U^H(a_t^p|h_t) \), which allows us to conclude that \( U^H(a_t^p|h_t) = \frac{1}{4r} \frac{rL + \phi H}{1 + \phi} \). Setting any price other than \( a_t^p \) results in \( p(h_t+dt) = 0 \) (by (NDOC-P)), and hence yields value of at most \( \frac{L}{4r} < \frac{1}{4r} \frac{rL + \phi H}{1 + \phi} \). Therefore, pooling is indeed optimal for \( H \). All of the above proves the following proposition.

**Proposition 2.** The following constitutes an equilibrium of the price signaling game for any profile of \( \lambda_t \). At any history \( h_t \):

1. if \( p(h_t) = \delta_\theta \) then all types play \( \theta \) and \( p(h_{t+dt}) = p(h_t) \) for all \( h_{t+dt} \supset h_t \);
2. if \( p(h_t) \in (0, 1) \) then:
   (a) type \( \theta = H \) plays \( a_t^p \) as given by (4) with probability one;
   (b) type \( \theta = L \) plays \( a_t^p \) w.p. \( 1 - \lambda_t \) and \( a_t^p|L = \frac{L}{2} \) w.p. \( \lambda_t \);
   (c) belief \( p(h_t \cup (a_t^p, \emptyset)) \) is computed according to (5); \( p(h_t \cup (a_t^p, H)) = 1 \), and \( p(h_t+dt) = 0 \) for all other \( h_{t+dt} \supset h_t \).

---

17Such an integer constaint will also apply to the bliss action \( a_t^{LH} \), but this would not affect the overall argument.
Many Types

The equilibrium described in the Proposition 2 translates immediately to the case with \(|\Theta| > 2\). In that case all types \(\theta \in \Theta \setminus L\) would behave as the high type above. Therefore, signaling through attrition is still possible. We now show that Theorem 2 can be applied in this problem as well to verify that payoff-relevant signaling is possible only through attrition, even despite the presence of informative reviews.

To do so, we need to verify that (SC) holds for the seller’s payoff function. Whenever a review arrives – which happens with probability \(1 - e^{-\phi dt}\) in every period – the continuation play is trivial (cf. Lemma 2 in the Appendix). Every type sets a myopically optimal price, thus obtaining payoff \(\theta/4\) per period. Therefore, we only need to consider strategies that are non-trivial at histories with non-degenerate beliefs \(p(h_t)\). The agent’s value from following a given strategy \(a\) starting from any such history \(h_t\) in the reduced game is given by

\[
U^\theta(a|h_t) = \sum_{s \in T, s \geq t} e^{-(r+\phi)(s-t)} \left[ e^{-\phi dt} \cdot a_t \left(1 - \frac{a_t}{E_s[\theta|a]}\right) + dt + (1 - e^{-\phi dt}) \cdot \frac{\theta}{4r}\right],
\]

where \(E_s[\theta|a] \equiv E[\theta|h_t \cup a_{(t,s)}]\). It is easy to see that this value function satisfies (SC): for any \(a', a''\) we have

\[
\mathcal{U}(\theta) \equiv U^\theta(a''|h_0) - U^\theta(a'|h_0)
\]

\[
= \sum_{s \in T, s \geq t} e^{-(r+\phi)(s-t)-\phi dt} \left[a''_t \left(1 - \frac{a''_t}{E_s[\theta|a'']}\right) - a'_t \left(1 - \frac{a'_t}{E_s[\theta|a']}\right)\right] dt,
\]

which is independent of \(\theta\). Therefore, the conclusions of Theorem 2 apply, and payoff-relevant signaling is only possible through attrition.

Takeaways

The price signaling model presented in this section does, despite being highly stylized, demonstrate that:

1. informative price signaling is possible without commitment, cost advantages, and with or without consumers learning from experiences;
2. signaling price may be either low (e.g., in the form of free trials or frequent sales) or inefficiently high (excluding most consumers) as a result of sunspots;
3. multiple informative equilibria exist that differ in the speed of separation;
4. due to the above, the empirical identification of price signaling in a given market is a daunting task.

Conclusion

This paper explores a model of dynamic signaling without commitment. In this model a single privately-informed agent takes an action every period, but cannot commit to future actions. The receiver tries to infer the agent’s information from his actions, and the receiver’s opinion is relevant to the agent’s payoff. The existing literature has assumed signaling to be impossible in such setting,
unless strong assumptions about off-equilibrium-path beliefs are adopted. This paper overturns this view, demonstrating that signaling is, in fact, possible even under reasonable off-path beliefs.

Contrary to the literature, we allow for suggestive signaling rather than requiring conclusive separation. We show that such signaling must necessarily happen through the attrition of the lowest type of the agent. In this attrition scenario, all types pool on the same action (or split across some payoff-equivalent actions), while the lowest type also plays some separating action with positive intensity.

The paper also contains a methodological contribution. In particular, we demonstrate the importance of seemingly innocuous assumptions regarding the type space for the conclusions one obtains. In particular, our results imply that perfect learning is possible in the limit as \( t \to \infty \) in the model with two sender types but not possible with finitely many types, while the literature demonstrates that perfect learning is possible with a continuum of sender types.

Finally, we explore an application of our results to a model of dynamic price signaling. We construct an informative equilibrium in which prices set by the firm contain information about the quality of its product. We show that price signaling can happen through both inefficiently low and inefficiently high prices, thus reconciling some of the disparate conclusions in the literature and arguing that empirical identification of price signaling in the data is a complicated venture.

References


G. Nöldeke and E. van Damme. Switching away from probability one beliefs. mimeo, 1990b.


Appendix

A Proofs: Preliminaries

Our first observation states that once there is no need for signaling any more – i.e., when the receiver’s belief assigns probability 1 to some type of the agent – there are no reasons for the agent to steer away from the myopically optimal action.

Lemma 2. In any equilibrium that satisfies (NDOC), if \(|S(h_t)| = 1\) then for all \(\theta\) and all \(h_s \supseteq h_t\):
\[
\alpha^\theta (B(\delta_{S(h_t)}(\theta)) \mid h_s) = 1.
\]

Proof. By (NDOC), for all \(h_s \supseteq h_t\): \(p(h_s) = \delta_{S(h_t)}\). In particular, \(p(h_s)\) is independent of all actions and outcomes during \([t, s)\). Therefore, the solution to \((\ref{eq:myopic})\) is given by pointwise maximization of the flow utility.

\(\square\)
Lemma 2 above is the direct consequence of (NDOC): actions cannot affect a degenerate belief under this assumption, hence the myopic optimum is chosen. This captures the main tension between signaling and dynamic concerns: given that no signaling can happen after separation, all costly informative actions should have taken place before the separation. However, if only the higher types attempt to follow the costlier action path in the first place – so that they can separate from the lower types – then separation, conceptually, has to happen at the very first period, when the agent decides between a “high-cost” and a “low-cost” path. The two arguments together imply that the action path that leads to the separation must have zero length – which, of course, implies that it cannot be separating since it is effectively costless to mimic. The remaining statements formalize this intuition. However, before getting to them we use Lemma 2 to prove Lemma 1 from the text.

Proof of Lemma 2. Parts 1 and 2 are trivial. For part 3, denote the original equilibrium strategy and belief profile as $\alpha_1$ and $p_1$ respectively. Construct the new equilibrium $(\alpha_2, p_2)$ by setting $\alpha_2(h_t) = \alpha_1(h_t)$ and $p_2(h_t) = p_1(h_t)$ for all on-path histories $h_t$. Then for all off-path histories $h_t$ set $p_2(h_t) = \delta_{S}(h_t)$ where $h \subset h_t$ is the last on-path history preceding $h_t$. The strategy $\alpha_2(h_t)$ for off-path histories $h_t$ can be chosen arbitrarily as long as it conforms with Lemma 2.

Belief profile $p_2$ will then satisfy (NDOC-P) and be consistent with the strategy $\alpha_2$. The strategy itself will be optimal at off-path histories by Lemma 2. Optimality of $\alpha_2$ at any on-path history $h_t$ can be verified by observing that $U^a(h_t \cup a)$ is the same in both equilibria for all $a \in A(h_t)$ and smaller in the newly constructed equilibrium for $a \in A \backslash A(h_t)$. I.e., the choice between any pair of off-path actions is unaffected by the off-path modifications, while deviations to off-path actions are less appealing in the new equilibrium. Therefore, $(\alpha_2, p_2)$ is an equilibrium.

B Proofs: Two Types

Proof of Theorem 1. Statement 1. Suppose first that there exist $h_t \in H$ and $a \in A$ such that $\alpha^H(a|h_t) > 0$ but $\alpha^L(a|h_t) = 0$. Then $p(h_t \cup (a,x)) = \delta_H$ for any $x \in X$ by (NDOC). By playing $a$ at $h_t$ the low type receives the highest possible continuation utility after $t$ (since by Lemma 2 he can play the myopically optimal action thereafter), while by following the equilibrium path he receives strictly less. The utility is bounded, hence for $dt$ small enough deviating to $a$ at $h_t$ is optimal for $L$.

Payoff-equivalence is shown as follows: for any two $a, a' \in A$ such that $\alpha^H(a|h_t) > 0$ and $\alpha^H(a'|h_t) > 0$ it must be that $U^H(h_t \cup a) = U^H(h_t \cup a')$, otherwise the high type would only play one of the actions and not the other. The first part of the argument showed that $\alpha^L(a|h_t) > 0$ and $\alpha^L(a'|h_t) > 0$, hence the same applies to the low type as well.

Statement 2. Begin with the first part (that $a \in B(\delta_L|L)$). For any such $a$ that $\alpha^H(a|h_t) = 0$ and $\alpha^L(a|h_t) > 0$ and any outcome $x$, we have $p(h_{t+dt}) = \delta_L$, where $h_{t+dt} = h_t \cup (a,x)$. By Lemma 2 $a' \in B(\delta_L|L)$ must be played at all histories beginning with $h_{t+dt}$. If $a \notin B(\delta_L|L)$ then playing $a'$ at $h_t$ instead – and continuing with $a'$ at all subsequent histories – yields a strictly higher flow payoff at $h_t$ and the same continuation payoff. Hence playing $a$ at $h_t$ was not optimal.

The second part of the second statement follows from the same argument as did payoff equivalence for $L$ in the first statement.

C Proofs: Finite Types

From this point onwards outcomes $x$ are assumed to be uninformative. Lemma 2 continues to hold in this setting. Before stating the proof of Theorem 2 we need some supplementary lemmas. We begin by
arguing in Lemma 3 that at no history can actions lead to separation of types into disjoint sets that can be compared by a strong set order – unless one of these sets is a singleton coinciding with the lower bound of the other set. In particular, we show that sets of types in the support of two different actions have to necessarily overlap (not in the sense of having common elements, but in the sense of order on Θ).

Lemma 3. Suppose (MON) holds and dt is small enough. Fix any equilibrium and any history h_t. Then for any a', a'' ∈ A(h_t) we have \( S(h_t ∪ a') ≥ S(h_t ∪ a'') \), with equality only if \( S(h_t ∪ a') \) is a singleton.\(^{18}\)

Proof. Assume by contradiction that \( S(h_t ∪ a') < S(h_t ∪ a'') \) for some \( a', a'' ∈ A(h_t) \). Pick any type \( θ ∈ S(h_t ∪ a') \) and any strategy \( a' \) on path for \( θ \) at \( h_t \). Then deviating to \( a'' \) at \( h_t \) and following \( a' \) after \( t \) is strictly better for \( θ \) than following \( a' \) throughout. To see this, recall that \( u^θ \) is increasing in \( p_θ \) by (MON) – and reputation \( p_θ ≥ δ S(h_t ∪ a'') \) generated by the deviation for all \( s > t \) is strictly higher than any reputation on equilibrium path, since \( S(h_t ∪ a'') > S(h_t ∪ a') \).\(^{19}\) This contradicts \( a' \) being optimal for \( θ \) as long as \( dt \) (period length and, hence, utility weight on the current period) is small enough.

Now suppose \( S(h_t ∪ a') = S(h_t ∪ a'') \). Suppose by way of contradiction that \( |S(h_t ∪ a')| > 1 \), meaning \( S(h_t ∪ a') < S(h_t ∪ a'') \). Then among all types in \( S(h_t ∪ a') \) there exists such \( θ \) that receives reputation \( p_θ < δ S(h_t ∪ a'') \) with positive probability for all \( s > t \) (this follows from belief consistency). Such \( θ \) would strictly benefit from the deviation described in the first part of this proof, yielding a contradiction.\(^{20}\)

Lemma 4 below is an intuitive extension of the monotonicity of the optimal action w.r.t. type (“higher types take higher actions”) from actions in the static model to strategies in dynamics. It relies heavily on (SC) property. The main problem in the dynamic setting is the lack of any nice complete order over strategies \( a \), so given two arbitrary strategies, we generally cannot say which one of them is “higher”. Therefore, we rephrase monotonicity to say instead that if a given strategy (or its equivalent) is optimal for two agent types, then it must also be optimal for all types in between. We cannot say with certainty that the given strategy is chosen on equilibrium path by any of these types in between, but we can claim that any profile they choose must be payoff-equivalent to the one under consideration.

Lemma 4. Suppose (SC) holds. Fix any equilibrium and history \( h_t \). If there exists a pair of strategies \( a, a' ⊃ h_t \) that are payoff-equivalent at \( h_t \) and are on path at \( h_t \) for some types \( \hat{θ} \) and \( \hat{θ} > \hat{θ} \) respectively, then any strategy \( \tilde{a} ⊃ h_t \) on path at \( h_t \) for any \( \hat{θ} ∈ (\hat{θ}, \hat{θ}) \) must be payoff-equivalent at \( h_t \) to \( a, a' \).

Proof. Fix any such \( \tilde{a} \). Strategy \( \tilde{a} \) has to be optimal for type \( \hat{θ} \). In particular, when evaluated at \( h_t \), it has to be better than \( \tilde{a} \):

\[
U^{θ}(\tilde{a}|h_t) ≥ U^{θ}(\hat{a}|h_t).
\]

The same holds for type \( \hat{θ} \), since \( \tilde{a} \) and \( a \) are payoff-equivalent:

\[
U^{θ}(\tilde{a}|h_t) = U^{θ}(a|h_t) ≥ U^{θ}(\hat{a}|h_t).
\]

At the same time, \( \hat{θ} \) at least weakly prefers \( \tilde{a} \) to \( \hat{a} \), meaning that the converse holds for \( \hat{θ} \):

\[
U^{θ}(\hat{a}|h_t) ≤ U^{θ}(\tilde{a}|h_t).
\]

\(^{18}\)This Lemma and the remainder of the Appendix uses the notation \( S(h_t ∪ a) \equiv S(h_t ∪ (a, x)) \) for all \( x \in X \) in the support. This object is well-defined in equilibrium for on-path histories and actions because the support of \( x \) is type-independent and equilibrium beliefs must be consistent. We are adopting the simplifying assumption that the same holds off the equilibrium path, but this is not necessary for the arguments to go through as long as (NDOC) holds.

\(^{19}\)We are also using the fact that \( u^θ \) is independent of outcomes.
If this inequality is strict, then this is a direct contradiction with (SC), which requires that $U^\theta(\mathbf{a}|h_t) - U^\theta(\tilde{\mathbf{a}}|h_t)$ as a function of $\theta$ either crosses zero once, or is exactly zero. \hfill \Box

Lemma 5 below is crucial for our argument in that it bridges the gap between actions and strategies. In particular, it shows that if two types of the agent choose the same action at time $t$ then there should also exist a strategy (possibly together with its equivalent) that is optimal for both types.

**Lemma 5.** Suppose (MON) and (SC) hold and $dt$ is small enough. Fix any equilibrium and any $h_t \in H$. There exist $h_t$-payoff-equivalent strategies $\tilde{\mathbf{a}}, \mathbf{a} \supset h_t$, on path at $h_t$ for $S(h_t)$ and $\bar{S}(h_t)$ respectively.

**Proof.** We will proceed by induction on the support size $|S(h_t)|$. The claim of the lemma holds trivially for $|S(h_t)| = 1$, and by Theorem 1 it also holds for $|S(h_t)| = 2$. The remainder of the proof shows that if the claim holds when $|S(h_t)| = k - 1$ then it also holds when $|S(h_t)| = k \geq 3$. Let $x : H \to X$ denote an outcome profile which prescribes some outcome for every history. Fix some $x$. Coupled with some pure strategy and the equilibrium belief system $p$, it fully determines the path of play and the agent’s payoffs. One such path is constructed below.

Begin the second layer of induction, iterating forwards on time periods from $t$. At $h_t$ and any subsequent history $h_s \supset h_t$, one of the following must apply:

1. There is an action $a$ on path for both types $\bar{S}(h_t)$ and $\bar{S}(h_t)$ at $h_s$. If this is the case, call $h_s$ a non-splitting history and continue to $h_{s+dt} = h_s \cup (a, x(h_s))$.

2. There is no action $a$ on path for both $\bar{S}(h_t)$ and $\bar{S}(h_t)$ at $h_s$. If this is the case, call $h_s$ a splitting history.

Proceed along the non-splitting path (according to the chosen $x$) until the first splitting history $h_s$. Pick arbitrary actions $\tilde{a}$ and $\mathbf{a}$ that are on path for $\bar{S}(h_t)$ and $\bar{S}(h_t)$ at $h_s$ respectively, and consider two continuation histories $\bar{h}_{s+dt} \equiv h_s \cup (\tilde{a}, x(h_s))$ and $\bar{a}_{s+dt} \equiv h_s \cup (\mathbf{a}, x(h_s))$. Then we have that $|S(h_{s+dt})| < |S(h_s)| = k$ for both continuation histories, because $S(h_{s+dt}) \subseteq S(h_s) \cup S(h_s)$ and $\bar{S}(h_{s+dt}) \subseteq S(h_s) \cup \bar{S}(h_s)$. Therefore, by the induction assumption, the statement of the lemma holds at both $\bar{h}_{s+dt}$ and $\bar{a}_{s+dt}$.

In particular, statement of the lemma for $h_{s+dt}$ states that there exist two $\bar{h}_{s+dt}$-payoff-equivalent strategies on path at $\bar{h}_{s+dt}$ for $S(h_s)$ and $\bar{S}(h_{s+dt})$ respectively. Playing $\tilde{a}$ at $h_s$ is on path for both of these types, hence there also exists a pair of strategies on path at $h_s$ for the two types respectively, which grant the same payoff conditional on $x$. \footnote{It does not matter for our argument if all types assign probability zero to outcome $x(h_s)$ conditional on $\tilde{a}$.} However, the argument above applies to any outcome profile $x$ and in particular to any outcome $x(h_s)$, hence there also exists a pair of strategies $\tilde{a}', \tilde{a}''$ on path at $h_s$ for the two types $S(h_s)$ and $\bar{S}(h_{s+dt})$ respectively, which are payoff-equivalent at $h_s$ (unconditionally).

By a mirror argument, there also exist $h_s$-payoff-equivalent strategies $\tilde{a}', \tilde{a}''$ on path at $h_s$ for $\bar{S}(h_s)$ and $\bar{S}(h_{s+dt})$ respectively. Note further that by Lemma 3 we have that $\bar{S}(h_{s+dt}) > \bar{S}(h_{s+dt})$. Lemma 4 hence applies: $\tilde{a}''$ must be payoff-equivalent to $\tilde{a}', \tilde{a}''$, thus so is $\tilde{a}'$. We have shown that the statement of the lemma holds at $h_t$ if $|S(h_t)| = k$ and $h_t$ is a splitting history.

We are left to cover non-splitting histories. Suppose $h_t$ is non-splitting. Fix $x$. Then we know that the statement of the lemma holds at the first splitting history $h_s$ following $h_t$ along the path of pooling actions and fixed outcomes $x$. Therefore, there exists a pair of strategies on path at $h_t$ for $\bar{S}(h_t)$ and $\bar{S}(h_t)$, which grant the same payoff at $h_t$ conditional on $x$. This applies to any outcome profile $x$, hence there exists a pair of strategies $\tilde{a}, \mathbf{a}$ on path at $h_t$ for $\bar{S}(h_t)$ and $\bar{S}(h_t)$, which are payoff-equivalent at $h_t$. This concludes the induction argument and the proof of the lemma. \hfill \Box
Proof of Proposition 7. Let $\theta \equiv S(h_t)$. Note that the statement of the proposition holds trivially if $|S(h_t)| = 1$, so for the remainder of this proof we assume that this is not the case (i.e., $\theta \neq \bar{\theta}$). From Lemma 3 we know there exist $h_t$-payoff-equivalent $\bar{a}, a \supset h_t$ on path at $h_t$ for $\theta$ and $\bar{\theta}$ respectively. Then by Lemma 4 any pure strategy $a \supset h_t$ on path at $h_t$ for any $\theta \in S(h_t) \setminus \{\bar{\theta}, \bar{\theta}\}$ is payoff-equivalent at $h_t$ to $\bar{a}, a$.

Suppose now there exists a pure strategy $a' \supset h_t$ on path at $h_t$ for $\bar{\theta}$, which is payoff-distinct at $h_t$ from $a$. By (SC), all types $\theta \in S(h_t) \setminus \bar{\theta}$ must have a strict preference at $h_t$ between $a$ and $a'$. The former is optimal for these types, hence $a'$ is only on path for $\bar{\theta}$. The two strategies cannot prescribe different actions at $h_t$ in equilibrium – $a(h_t) \neq a'(h_t)$ – since this is in violation of Lemma 3. The same, however, applies to any subsequent history, hence $a(h_s) = a'(h_s)$ for all $h_s \supset h_t$. This contradicts $a$ and $a'$ being payoff-distinct, hence such $a'$ does not exist. Therefore, any pure strategy $a$ on path at $h_t$ that is $h_t$-payoff-distinct from $a$ is only on path for $\bar{\theta}$. This concludes the proof.

Proof of Theorem 3. From Proposition 4 all $a \in A(h_t)$ are on path for $\bar{\theta}$, which proves the first statement of the theorem.

From the fact that payoff-relevant signaling happens at $h_t$ we know that there exist two pure strategies $a, a'$ that are payoff-distinct at $h_t$ and prescribe different actions at $h_t$: $a \equiv a_t \neq \bar{a} \equiv a_t$. From Proposition 8 we know at least one of these strategies – suppose $a$ is on path for $\bar{\theta}$ but not for any other $\theta \in S(h_t) \setminus \bar{\theta}$ at $h_t$. Furthermore, it follows from the definition of payoff-relevant signaling that there is no $a' \supset h_t \cup a$ that is payoff-equivalent to $\bar{a}$ at $h_t$. Therefore, $a$ is only on path for $\bar{\theta}$, while $\bar{a}$ is optimal for all $\theta \in S(h_t)$ at $h_t$.

We now show that $a \in B(\delta_0[\bar{\theta}]).$ If this is not true then type $\bar{\theta}$ can play some $a \in B(\delta_0[\bar{\theta}])$ at $h_t$ and every history after it. Compared to following $\bar{a}$, this strategy would yield the same payoff at all times $s > t$ and a strictly higher payoff at $t$ (by the same argument as in the proof of Theorem 4), hence $a$ is not optimal for $\bar{\theta}$ at $h_t$ – a contradiction.

To complete the proof of statements 2 and 3 of the theorem, we need to show that $\bar{a} \notin B(\delta_0[\bar{\theta}]).$ Assume not. Consider the strategy of playing $\bar{a}$ at $h_t$ and all subsequent histories. Compared to following $a$, this strategy would yield $\bar{\theta}$ a weakly higher payoff at all times $s > t$ and a strictly higher payoff at $t$ (due to $p(h_t \cup a) > \bar{a}$ and the strict part of (MON)), hence $\bar{a}$ is not optimal for $\bar{\theta}$ at $h_t$ – a contradiction.

This completes the proof of Theorem 2.

Proof of Corollary 4. The statement is proved in the text. The low type must be indifferent between taking a separating action $\bar{a}$ at $h_t$ and pooling on $a$ at $h_t$ and separating at $h_{t+d_t}$. This indifference dictates that one period of pooling must be exactly as attractive as one period of being revealed as $\bar{\theta}$, i.e., $\mathbb{E}[u^\bar{\theta}(\bar{a}, p(h_t \cup (\bar{a}, x))) | \bar{\theta}] = u^\bar{\theta}(a, \delta_0)$. 

Proof of Corollary 2. Proposition 1 states that all pure strategies $a'$ on path at $h_t$ for any $\theta \in S(h_t) \setminus S(h_t)$ are payoff-equivalent at $h_t$. Since there is no payoff-relevant signaling in equilibrium, the set of such strategies is a singleton: if there is more than one then there exists $h'_t \supset h_t$ at which the two prescribe different actions, but that constitutes payoff-irrelevant signaling at $h'_t$ (the two strategies coincide on $[t, t')$, hence they are payoff-equivalent at $h'_t$).

Therefore, at any $h_t$ there exists some $\bar{a} \in A$ such that $\alpha^\bar{\theta}(h_t)(\bar{a}) = 1$ for all $\theta \in S(h_t) \setminus S(h_t)$. Together with part 1 of Theorem 3 this means that $S(h_t \cup \bar{a}) = S(h_t)$. By part 3 of the theorem, $\bar{a}$ is the unique element of $\bar{a} \in A(h_t) \setminus B(\delta_0[\bar{\theta}])$. By part 2 of the theorem, for any $a \in A(h_t) \cap B(\delta_0[\bar{\theta}])$ we have $S(h_t \cup a) = S(h_t)$. Since all on-path histories $h_{t+d_t}$ can be written as $h_{t+d_t} = h_t \cup (a, x)$ for some $a \in A(h_t)$

21To be slightly more precise, the argument applies to any $h_s$ such that $|S(h_s)| > 1$. Otherwise Lemma 2 kicks in and implies that all pure strategies on path at $h_s$ are $h_s$-payoff-equivalent.
and $x \in X$, and outcomes $x$ do not change support $S$, we obtain that for any pair of on-path histories $h_t, h_{t+dt}$: $S(h_{t+dt}) \in \{S(h_t), [S(h_t)]\}$. Applying this observation iteratively from $h_0$ (for which $S(h_0) = \Theta$) completes the proof. \qed