Experts, Quacks and Fortune-Tellers: Dynamic Cheap Talk with Career Concerns*

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Abstract

The paper studies a dynamic communication game in the presence of adverse selection and career concerns. An expert of privately known competence, who cares about his reputation, chooses the timing of the forecast regarding the outcome of some future event. We find that in all equilibria in a sufficiently general class earlier reports are more credible. Further, any report hurts the expert’s reputation in the short run, with later reports incurring larger penalties. Reputation of a silent expert, on the other hand, gradually improves over time.

Keywords: Career concerns, reputation, dynamic games, games of timing, strategic information transmission.

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1 Introduction

Where there is uncertainty, there are analysts – be it stock prices, elections, or sports matches. Any major event summons numerous predictions of its outcome from people who claim to be experts in the field. However, not all of these predictions are necessarily backed by knowledge or understanding of the situation. This has become even more true in the Internet era when any quack can start a public blog at near-zero cost to broadcast his thoughts and opinions even if they have no insight into the topic. They enter the predictions race for the very same reasons that competent experts do – most often this is a chase for reputation and associated benefits. The unfounded predictions made by quacks then generate nothing but noise, hindering the transmission of potentially valuable information from the real experts to the public.

The main tension in this situation comes from the fact that even when an expert claims to have a reputable source of information, these claims are often difficult (if not impossible) to verify. This non-verifiability can harm both sides of the market. On the one hand, it means that the public cannot distinguish a justified prediction from quacks’ random utterances. On the other hand, it makes it difficult for the competent experts to signal their competence and gain deserved reputation. The fact that even well-grounded forecasts are not that easy to communicate to the audience in such environment is illustrated by the story of Nouriel Roubini: he sounded like a madman to fellow economists in 2006 when he was predicting a housing bust and a deep recession; a year later he was considered a prophet.[1]

How credible should experts’ predictions be in the eyes of the public? Can competent experts transmit their private information through the noise? How does expert’s reputation react to predictions that are not immediately verifiable? Does the timing of a prediction matter for its perception by the public? Our paper tries to answer these questions, with a particular emphasis on timing. In our model an expert, who is privately aware of his competence, makes a choice of whether and when to make a prediction about the outcome of some future exogenous event (state of the world). A competent expert (a maven) may have some private knowledge about the outcome, while an incompetent expert (a quack) never does. There is no direct conflict between the expert and the observer (the public) in our model: the expert only cares about his reputation, while the observer only cares about the information concerning the outcome. The conflict comes from within the market of experts, with quacks trying to blend in with mavens in pursuit for reputation (and benefits that high reputation grants), preventing mavens from conveying valuable information to the public.

The key feature of our model is timing. The competent expert obtains his private knowledge at an uncertain time. Thus, on the one hand, an early prediction is less likely to be backed up by private knowledge and should be perceived by the public as less reliable than a later prediction. On the other hand, if this is the case then quacks should prefer to report later, thus devaluing late predictions. We discover that in the end, if predictions are made at multiple dates in equilibrium, then later predictions are less informative. This comes from two facts. First, if maven never makes a prediction at date \( t \) then a quack does not either, since quack’s only goal is pretending to be maven. Second, maven only makes a prediction today if doing so is better than waiting and using

[1] More details can be found in this NYT article: http://www.nytimes.com/2008/08/17/magazine/17pessimist-t.html
his information to make a prediction tomorrow – i.e., if tomorrow’s report is valued less by the public because it is less informative.

Later reports in such equilibria, where reports are made at multiple dates, are less informative because they are more likely to have been made by a quack than by a maven. Therefore, reporting late affects expert’s short-run reputation (until the state is revealed) more adversely than making a report early. A more surprising finding that, as we show, follows from this logic is that all predictions in such equilibria, although considered informative, are received with solid scepticism in the sense that the expert’s reputation drops once he makes a prediction. Thus silence is indeed gold in our model – silent experts see their reputation gradually improving. Those, on the other hand, who choose to make a prediction and take a hit to their reputation, are gambling for the grand prize that is the reputation bonus for predicting the outcome correctly.

A typical dynamics of expert’s reputation arising from our model is illustrated in Figure 1. In this example the event occurs in period 6 and the expert starts with reputation $b_0$. The expert makes his report in period 4, and until that his reputation gradually increases. After the report his reputation drops until the event outcome is revealed, at which point he receives a reputation premium if his prediction turned out correct and is penalized by low reputation otherwise.

Given everything said above, it is not obvious why a quack would ever prefer to make any prediction, i.e., take a risky gamble at the cost of short-run reputation, when staying silent would yield a risk-free high reputation. As we show, equilibria of the form above (where reports are made at multiple dates) only exist if experts are sufficiently risk-loving or, equivalently, if gains from reputation are sufficiently convex – i.e., if the gamble of making a report is appealing enough to the quack. Whenever this is not the case, only “static” equilibria exist, in which all reports are made at some single predetermined date.

Finally, given the multiplicity of equilibria, we also provide a comparison of different equilibria from the observer’s point of view (i.e., in the sense of the amount of information conveyed by the reports). We show that “extending the deadline” in the sense of allowing later predictions to be made is beneficial. We also show that a maven making predictions that are not supported by any

\[2\]In one special case there also exist degenerate equilibria, in which quack never makes any predictions for the fear of being proved wrong. See Section 5.2 for details. Furthermore, equilibria with informative communication are implied here, since equilibria with uninformative communication (babbling) only always exist.
private information — and are thus as uninformative as those of a quack — may in fact improve the overall quality of information transmission.

The paper is organized as follows. Section 2 contains a review of the relevant literature. In Section 3 we formulate the model. The main results are presented in Section 4. Section 5 contains extensions and alternative specifications. Section 6 concludes. All proofs are relegated to the Appendix.

2 Literature Review

The current paper mainly contributes to two strands of literature: communication with career concerns and the timing of communication.

The importance of experts’ career concerns for informative communication was first argued by Holmström [1999]. One of Holmström’s original examples illustrates that an expert endowed with imprecise private information may be too reluctant to utilize it for the fear of making a mistake and appearing incompetent. The literature that followed has extended this result, showing that indeed in various settings, in the presence of career concerns, an informed expert at least partially ignores his private information and prefers to herd with public information or reports of other experts (see, for example, Scharfstein and Stein [1990], Trueman [1994], Ely and Välimäki [2003], Ottaviani and Sørensen [2006a], Dasgupta and Prat [2008]).

Other papers have shown that some cohort of analysts — or even all of them in some settings — may, conversely, resort to extreme reports, overstating their private signals in order to separate themselves from “herders” (see Prendergast and Stole [1996], Graham [1999], Hong, Kubik, and Solomon [2000], Lamont [2002], Ottaviani and Sørensen [2006b], Mariano [2012]). Either way, it is generally agreed that experts’ career concerns make information transmission noisy.

Of all papers mentioned above only a few look at the dynamics of announcements. In the model of Prendergast and Stole [1996] the expert obtains his private information gradually over time, and his competence determines the speed of learning. They establish that the experts overreact to early pieces of information, while later on they become too reluctant to change their decisions and thus underreact to late information. Predictions of a model by Graham [1999] can be interpreted in a similar way. Curiously, Hong, Kubik, and Solomon [2000] and Lamont [2002] find a completely opposite pattern in the data: older and more established experts usually make more extreme predictions. However, timing of the prediction or a decision is never a variable of choice for the expert in these papers as it is in our paper.

Keskek, Tse, and Tucker [2014] show empirically that competent experts tend to make their reports earlier — so earlier reports are more informative and are perceived more favorably, — and explain this through preemption mechanisms. We show that competition is not necessary for this phenomenon to arise.

Zábojnik [2001], Ely and Välimäki [2003] and Klein and Mylovanov [2017] argue that if all experts are patient enough then this noise vanishes and communication efficiency is restored.

Backus and Little [2018] show that making experts admit uncertainty (not knowing the answer) is also not a trivial problem in the presence of career concerns.
In all papers mentioned above, if the expert does not have decision rights then he does not directly care about the outcome, only about his reputation. A separate literature (including, but not limited to Sobel [1985], Bénabou and Laroque [1992], Morris [2001], Pavesi and Scotti [2017]) explores the trade-off that an expert of uncertain bias faces when trying to manipulate the decision-maker’s action in every period. The trade-off is between immediate gains from biased recommendations today and building up credibility for the long run. We abstract from such scenarios, focusing on the case of pure career concerns.

Effects similar to career concerns models can be demonstrated in communication settings where sender is only interested in affecting receiver’s decision, but sender’s deceit can be detected with positive probability. In both settings the sender suffers if his advice is discovered to be incorrect. See Dziuda and Salas [2017], Drugov and Troya-Martinez [2018] for examples of such communication models.

Our paper also contributes to the emerging literature on timing of information and using timing of the decision as a signaling device. Guttman, Kremer, and Śkrzyzacz [2014] provide a notable illustration on the importance of timing. In the context of dynamic disclosure, they show that the same piece of hard (verifiable) information can induce different reactions when disclosed at different times. In other related papers, Guttman [2010], Acharya, DeMarzo, and Kremer [2011], Aghamolla and An [2015], and Gratton, Holden, and Kolotilin [2018] also investigate optimal dynamic disclosure of verifiable information. In contrast to these papers, we deal with soft information, which cannot be credibly disclosed. Ottaviani and Sørensen [2006d] explore the timing of bets on the betting market. Grenadier, Malenko, and Malenko [2016] study a setting in which the informed expert uses timing of his report to manipulate the timing of the observer’s decision. Bobtcheff and Levy [2017] consider a real options problem, where an entrepreneur’s ability determines the amount of information she has about the project. The timing of entrepreneur’s request for funds in this setting signals both her ability and her belief about the project. This paper is similar to ours in spirit, but focuses on a distinct problem, where the structure of players’ payoffs is non-trivially different.

A separate strand of literature on dynamic cheap talk explores dynamic revelation of static information. It finds, to some surprise, that even if all agents possess all of their respective information in period zero, allowing for multi-period communication may sometimes allow for higher payoffs. For more details see Aumann and Hart [2003], Krishna and Morgan [2004], Alonso and Rantakari [2013], Chen, Goltsman, Hörner, and Pavlov [2017], and Lipnowski and Ravid [2017].

Our model is trivially different in that sender’s information acquisition is staggered over time. Experimentation and/or information acquisition with career concerns, as studied by Board and Meyer-ter-Vehn [2013], Aghion and Jackson [2016], Bonatti and Hörner [2017], Ben-Porath, Dekel, and Lipman [2018], Bhaskar and Thomas [2018], Cheng and Li [2018], and Halac and Kremer [2018], is another close strand of literature. Its primary focus is on how the expert designs his information flow – given that this decision may have direct payoff consequences for the expert and/or be observable to the principal, – rather than how he transmits the acquired information. Gerardi and Maestri [2012], Hidir [2015], Guo [2016], Halac, Kartik, and Liu [2016], and Häfner and Taylor [2018] explore this problem from the observer’s point of view, designing optimal mechanisms for experimentation or information acquisition.

Finally, our paper takes the market as given rather than designing it in such a way as to extract
the most information from the expert. A general approach to dynamic mechanism design when experts have evolving private information has been proposed by Pavan, Segal, and Toikka [2014]. A subfield of mechanism design explicitly deals with optimal statistical testing of experts’ competence (knowledge of a signal-generating process): see Al-Najjar and Weinstein [2008], Olszewski and Peski [2011] as examples and Olszewski [2015] for a recent survey. The most recent are the contributions by Di Pei [2016], Smolin [2017], and Deb, Pai, and Said [2018]. Our paper is different from this literature in that it does not give the observer the power to design payoffs or information feedback. Instead, it asks the question of whether market forces alone can enable informative communication.

3 The Model

3.1 Primitives

Time is discrete and finite: \( t \in \{0\} \cup T \) where \( T \equiv \{1, \ldots, T\} \) for some \( T > 0 \). An underlying standard probability space is implied throughout the paper. The probability measure on this space is denoted by \( P \).

State of the world. There is a binary state of the world \( \omega \) which can be either good or bad: \( \omega \in \{G, B\} \). The commonly held prior belief that the state is good is \( P(\omega = G) = p_0 \in [\frac{1}{2}, 1) \). Initially the state is uncertain; at the end of period \( T \) the state is revealed.

Players. There are two players: an observer (she) and an expert (he). Both players live for \( T \) periods and do not discount the future.

The observer has no actions in the model and serves only to form beliefs about the expert’s competence. In the discussion surrounding the model, we assume that she is interested in information about state. To fix ideas, one may think that the observer chooses a binary action from \( \{G, B\} \) at time \( T \) and receives a fixed reward if and only if her action matches the state – but we do not model this decision explicitly.

The expert has a binary type \( \gamma \in \{C, I\} \): he can be competent (C) or incompetent (I). We also call competent and incompetent expert as maven and quack respectively. The type is privately known by the expert, but is not known by the observer. The observer’s initial belief that the expert is competent is \( b_0 \in (0, 1) \).

At some random time \( t^* \sim F(t) \), which is not known to anybody, the competent expert observes a signal \( \eta^* \in \{G, B\} \) about the state, with precision \( \rho := P(\eta^* = G|\omega = G) = P(\eta^* = B|\omega = B) \). For most of the paper we assume \( \rho = 1 \), but in Section 5.3 we show that all results continue to hold in case \( \rho \in (\frac{1}{2}, 1) \) given some extra conditions. We assume that \( F(t) \) is a measure with full support on \( T \) and that \( F(T) < 1 \), i.e., there is a positive probability that the signal arrives at any time \( t \) and it is possible that it never arrives. We denote the conditional probability (hazard rate) of signal arrival in period \( t \) as \( \lambda(t) := \frac{F(t) - F(t-1)}{1 - F(t-1)} \).

The expert receives a per-period “reputation payoff” \( w\left(\frac{b_t}{1-b_t}\right) \) which depends on \( b_t \) – the observer’s belief about the expert’s competence held at the end of period \( t \). We assume \( w(x) \) to be strictly increasing in its argument. As a normalization, we let \( w(0) = 0 \). After the state is revealed, the expert receives a terminal payoff \( \theta \cdot w\left(\frac{b_T}{1-b_T}\right) \) for some \( \theta > 0 \), representing the expert’s continuation value from the reputation he has accumulated. Payoffs are interpreted as

5The assumption that continuation payoff is a scaled version of the flow payoff is made for notational convenience
coming from some external source rather than the observer directly. A highly regarded analyst can bargain higher wage from employers in the labor market, while all of the interested public acts as the observer in forming analyst’s reputation.

**Communication.** In any period \( t \in T \) the expert can send a report \( m \in \{G, B\} \) to the observer, indicating his prediction about state \( \omega \). The report is not verifiable, i.e., the expert’s private information is not ever observable and/or contractible. Additionally, we assume that the expert can send at most one report throughout the game.\(^6\)

### 3.2 Timing

At time \( t = 0 \), the state of the world \( \omega \) and the expert’s type \( \gamma \) are realized; expert’s private signal realization \( \eta^* \) and signal arrival time \( t^* \) are drawn from respective distributions. After that, in every period \( t \in \{1, ..., T - 1\} \) the stage game proceeds as follows:

1. If \( t = t^* \) and the expert is competent, he observes the realization of \( \eta^* \);
2. The expert updates his belief about the state conditional on observed \( \eta^* \) (if any) and decides whether to send a report \( m \in \{G, B\} \) to the observer;
3. The observer updates her beliefs about the state \( (p) \) and about the expert’s competence \( (b) \) conditional on the expert’s report or lack of thereof;
4. The expert receives payoff \( w \left( \frac{b}{1-b} \right) \);

In period \( T \) steps 1 and 2 take place as above, but instead of steps 3 and 4 the following happens:

3. State \( \omega \) is publicly revealed;
4. All players update their beliefs accordingly;
5. The expert receives a terminal lump-sum payoff \( \theta \cdot w \left( \frac{b}{1-b} \right) \).

### 3.3 Histories and State Variables

A *message history* is \( \mu_t = (m, s) \) if report \( m \) has been made in period \( s \leq t \) and \( \mu_t = \emptyset \) otherwise. A *public history* \( h^p_t \) is a tuple consisting of the variables that are publicly observable at the beginning of period \( t \): \( h^p_t = (t, \mu_{t-1}) \). The expert possesses private information about his type and his private signal in addition to whatever is publicly known. We define a type-\( \gamma \) expert’s *private history* as \( h^\gamma_t = (h^p_t, \eta^\gamma_t, t^\gamma) \), where \( \eta^\gamma_t \) describes expert’s private information:

- \( \eta^\gamma_t = \emptyset \) if no signal was observed in period \( t \) or before,
- \( \eta^\gamma_t = G \) if signal \( \eta^* = G \) was observed in period \( t \) or before,
- \( \eta^\gamma_t = B \) if signal \( \eta^* = B \) was observed in period \( t \) or before.

only. All results continue to hold if we assume that continuation payoff is an arbitrary function \( w_c \left( \frac{b}{1-b} \right) \) that satisfies all requirements that we impose on \( w \).

\(^6\)This constraint should not be seen as restrictive since the expert receives at most one private signal by time \( T \).
Variable $t^\gamma$ indicates the arrival time of this information, with $t^\gamma = 0$ meaning no information has yet arrived. For quacks we have that $t^l = 0$ and $\eta^l = \emptyset$. For mavens these variables can be expanded as $t^C = t^* \cdot \mathbb{1}(t \geq t^*)$, and $\eta^C = \eta^*$ if $t \geq t^*$, and $\eta^C = \emptyset$ otherwise. Notably, values $(\eta^*_t, t^\gamma)$ are only nontrivial for the competent expert, thus the incompetent expert’s private histories are equivalent to public histories, and hereinafter we will treat them as such. We also let $-\eta$ and $-m$ denote the “opposites” of $\eta$ and $m$ respectively: e.g., if $\eta = G$ then $-\eta = B$.

3.4 Expert’s Problem

At any pair of histories $\bar{h}^C_t, \bar{h}^C_t$ which differ only in signal arrival times $\bar{t} < \bar{t}^* \leq t$, the strategy of the competent expert must be the same.

This assumption requires that after the private signal is observed, the competent expert’s reporting strategy does not depend on its arrival time $t^*$. One may think of this as the expert not remembering when he received the information (but the information itself is never forgotten). This restriction bans strategies like "send a report two periods after receiving a signal". This, however, should not be considered a loss of generality, as the timing of signal arrival is neither observable by anyone except the expert, nor payoff-relevant for any player, so can be seen as nothing more than expert’s private randomization device.

Amnesia together with the fact that only histories with $\mu_{t-1} = \emptyset$ involve non-trivial choice of message allow us to define strategies on the smaller space of tuples $(t, \eta)$ rather than on all private histories $h^*_t = (t, \mu_{t-1}, \eta^*_t, t^\gamma)$. Therefore, we introduce the expert’s behavioral strategy as $r^\gamma(m, t)$, which denotes the probability of expert $\gamma$ making report $m$ at time $t$ conditional on having private information $\eta = \eta^t$ and having not made any report prior to $t^\gamma$. Finally, denote $r^\gamma(m, t) := \mathbb{E}_\eta r^\gamma(m, t)$. It represents the hazard rate of report $(m, t)$ as perceived by the observer who does not possess the expert’s private information $\eta$ (but these objects are still conditional on the expert’s type).

Expert’s optimization problem is hence as follows: at every private history $h^*_t$ such that no report has yet been made ($\mu_{t-1} = \emptyset$) expert of type $\gamma \in \{C, I\}$ chooses a continuation reporting strategy $\{r^\gamma(m, s)\}_{s \geq t}$ as a solution to the following problem:

$$V^\gamma_{t, \eta} := \max_{r^\gamma} \mathbb{E} \left[ \sum_{s = t}^{T-1} w \left( \frac{b(h^p_s)}{1 - b(h^p_s)} \right) + \theta \cdot w \left( \frac{b(h^p_s)}{1 - b(h^p_s)} \right) \bigg| t, \eta, \mu_{t-1} = \emptyset \right]$$

subject to evolution of belief $b(h^p_s)$ described in the following subsection. The expectation is taken over all future histories. We also introduce a shorthand notation for expert’s continuation value.

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$^7$This is a game of perfect recall, hence by Kuhn’s theorem behavioral and mixed strategies are equivalent.
from making report $m$ at history $h^T_i$:
\[
W^\gamma_{t,\eta}(m, \tau) := \mathbb{E}\left[\sum_{s=t}^{T-1} w\left(\frac{b(h^T_s)}{1 - b(h^T_s)}\right) + \theta \cdot w\left(\frac{b(h^P_T)}{1 - b(h^P_T)}\right) \middle| t, \eta, \mu = (m, \tau)\right].
\]

With this notation we have that report $(m, t)$ is optimal at $t$ if and only if $V^\gamma_{t,\eta} = W^\gamma_{t,\eta}(m, t)$. Moreover, let us use $W^\gamma_{t,\eta}(\emptyset)$ to denote similar value from not making any report until the end of period $T$ (i.e., conditional on $\mu_T = \emptyset$). Finally, as incompetent expert never receives private signal, we suppress subscript $\eta$ when talking about $V^I_{t,\eta}$ and $W^I_{t,\eta}(m, \tau)$, and refer to these objects as $V^I_t$ and $W^I_t(m, \tau)$ respectively.

### 3.5 Beliefs

Two important characteristics of any public history $h^P_T$ are public beliefs about the type of the expert and about the state of the world, $b(h^P_T)$ and $p(h^P_T)$. Recall that $h^P_T = (t, \mu_{t-1})$. We will use this together with the structure of our model to introduce (with abuse of notation) the following labels for beliefs:

\[
b(m, t) := b(s, (m, t)) \\
b_t := b(t, \emptyset) \\
p(m, t) := p(s, (m, t)) \\
p_t := p(t, \emptyset)
\]

for all $s \geq t$. In this notation, $b(m, t)$ is the belief about the expert’s type held by the observer at any time $s \geq t$ conditional on report $m$ made at time $t$, and $b_t$ is the same belief held in the absence of any reports. The same applies for the observer’s belief about state. This notation is well defined because once a report has been made, both beliefs are frozen in place since no further information can be conveyed from the expert to the public.

Finally, we let $b^\omega(m, t)$ denote the belief about the expert’s type given a terminal history $h^P_T = (T, (m, t))$ and given that the state was revealed to be $\omega$.

### 3.6 Equilibrium Definition

We are looking for Weak Perfect Bayesian Equilibria of the game, which consist of a strategy profile $(r^C G, r^C B, r^I G, r^I B)$ and a belief profile $(b(h^P_T), p(h^P_T))$ such that:

1. strategies $r^\gamma_{\eta}$ solve [1] given the observer’s updating rule for $b(h^P_T)$,
2. all players update their beliefs via Bayes’ rule on path.

We further adopt three following refinements (in addition to restriction to amnesiac strategies):

**OP** Off-path Pessimism: off the equilibrium path the beliefs are $p = p_0$ and $b = 0$, with the exception that extreme belief $b = 1$ is not updated;

**ML** Message Labeling: $r^C_G(G, t) \cdot r^C_B(B, t) \geq r^C_G(B, t) \cdot r^C_B(G, t)$ for all $t$;

**SY** Symmetry: $r^C_G(G, t) = r^C_B(G, t)$ and $r^C_B(B, t) = r^C_B(G, t)$ for all $t$. 

9
Off-path pessimism (OP) only makes it easier to sustain any given strategy profile as equilibrium because it makes deviations extremely unappealing for the expert. In particular, if there is some PBE with some off-equilibrium path beliefs then the same profile of strategies and on-path beliefs would still constitute a PBE when paired with off-path beliefs prescribed by (OP). The exception in (OP) comes into play only if $\rho = 1$ and only at histories at which report was supposedly made by an informed competent expert, but turned out to be incorrect. The exception says that expert is then still believed to be competent. This behavior of beliefs can be explained as the limiting case of the model as $\rho \to 1$.

Message labeling (ML) requires that report $m$ is more indicative of state $\omega = m$ than the other report. This assumption is without loss, since at any history $h^P_t$ we can assign message labels $G$ and $B$ to the two messages in such a way that (ML) is satisfied.

The only requirement that imposes any actual restrictions is symmetry (SY). It requires that the competent expert treats states and messages equally – if he has evidence of state $G$, he sends report $G$ with the same probability that he would have sent report $B$ if he had evidence of state $B$. This assumption is made for tractability, so that the observer’s belief about state stays at a constant level $p = p_0$ as long as no report is made.

4 Equilibrium Analysis

This section strives to characterize the set of all Weak PBE of the game. The main question that is answered in this section is as follows: assuming that in some equilibrium reports can only be made at some set of periods $S \subseteq T$, how do the expert’s strategies look and how does the informativeness of the reports change across different periods? It turns out that all equilibria have quite a lot of common structure. Proofs of all statements presented in this chapter can be found in the Appendix.

4.1 Belief Updating

This section specifies how exactly observer’s beliefs $b$ and $p$ evolve given the experts’ strategy profile $\{r^*_\eta(m, t)\}$.

Conditional on the expert not making a report, the observer’s beliefs are updated as follows:

$$\frac{b_t}{1 - b_t} = \frac{b_{t-1}}{1 - b_{t-1}} \cdot \frac{1 - r^C(G, t) - r^C(B, t)}{1 - r^I(G, t) - r^I(B, t)}.$$  (2)

Similarly, employing the Bayes’ rule we can derive the observer’s belief $b(m, t)$ following expert’s report $(m, t)$, and the observer’s terminal belief $b^\omega(m, t)$ given expert’s report $(m, t)$ and the realized state $\omega$:

$$\frac{b(m, t)}{1 - b(m, t)} = \frac{b_{t-1}}{1 - b_{t-1}} \cdot \frac{r^C(m, t)}{r^I(m, t)},$$

$$\frac{b^\omega(m, t)}{1 - b^\omega(m, t)} = \frac{b_{t-1}}{1 - b_{t-1}} \cdot \frac{\mathbb{E}_{\eta}[r^C_\eta(m, t)|\omega]}{r^I(m, t)}.$$  (3)

8Section 5.2 discusses alternatives to (OP) in case of perfect signals.

9Silence is informative about the state only if the competent expert conceals his private signals, and does so differently conditional on different information. Assumption (SY) explicitly prohibits the latter part of this.
4.2 Informativeness Measures

Conditional on no report by the end of period $T$ and realized state $\omega$, the observer’s terminal belief is $b^*(\emptyset) = b_T$, since silence is completely uninformative about state in symmetric equilibria, so revealing the state conveys no new information about the expert’s type.

Finally, another relevant belief is the observer’s belief about the current state, $p_t$. As mentioned before, symmetry implies that in the absence of the report this belief is frozen at its initial level, $p_t = p_0$. Following report $(m,t)$ the belief is updated as:

$$\frac{p(m,t)}{1-p(m,t)} = \frac{p_{t-1}}{1-p_{t-1}} \cdot \frac{(1-b_{t-1}) \cdot r^I(m,t) + b_{t-1} \cdot E [r_C^G(m,t)|\omega = G]}{(1-b_{t-1}) \cdot r^I(m,t) + b_{t-1} \cdot E [r_C^G(m,t)|\omega = B]}$$

$$= \frac{p_{t-1}}{1-p_{t-1}} \cdot \frac{1 + \frac{b_{t}G(m,t)}{1-b_{t}G(m,t)}}{1 + \frac{b_{t}B(m,t)}{1-b_{t}B(m,t)}} = \frac{p_{t-1}}{1-p_{t-1}} \cdot \frac{1 - b_{t}B(m,t)}{1 - b_{t}G(m,t)}. \tag{4}$$

4.2 Informativeness Measures

Given the strategies, we define the likelihood ratio of reports as

$$g(m,t) := \ln \left( \frac{b(m,t)}{1-b(m,t)} \right) - \ln \left( \frac{b_{t-1}}{1-b_{t-1}} \right) = \ln \left( \frac{r_C^G(m,t)}{r_C^I(m,t)} \right),$$

with $\pm \infty$ being admissible values. This ratio summarizes the information about the expert’s type contained in report $(m,t)$. If, for example, $g(m,t) < 0$, then report $(m,t)$ is more likely to have been made by a quack than a maven, so the expert’s reputation deteriorates upon making it. Since belief is a martingale, in case $g(m,t) < 0$ for both $m \in \{I,B\}$, the belief about the expert’s competence must improve if no report was made at $t$. The opposite happens whenever $g(m,t) > 0$, while if $g(m,t) = 0$ the report has no immediate effect on expert’s reputation – though it may still be affected by state revelation at $T$.

In addition to $g(m,t)$ (measure of informativeness about the expert’s type), we also introduce a measure of informativeness about state of the world:

$$i(m,t) := \ln \left( \frac{p(m,t)}{1-p(m,t)} \right) - \ln \left( \frac{p_{t-1}}{1-p_{t-1}} \right) = \ln \left( 1 + \frac{b_{t}G(m,t)}{1-b_{t}G(m,t)} \right) - \ln \left( 1 + \frac{b_{t}B(m,t)}{1-b_{t}B(m,t)} \right).$$

This measure shows how likely report $(m,t)$ is to be sent in state $G$ as opposed to state $B$. Positive values reinforce the observer’s belief in state $\omega = G$ after hearing this report, while negative values do the same for state $\omega = B$. Higher absolute values of $i(m,t)$ mean that more information is transmitted by message $(m,t)$ to the observer, meaning that belief $p(m,t)$ moves further away from $p_{t-1}$. Our results are not specific to the particular functional form of $i(m,t)$ and are compatible with any other measure of distance between $p(m,t)$ and $p_{t-1}$.
4.3 Supports of the Reporting Times

Given $\gamma \in \{C,I\}$ and $m \in \{G,B\}$, define support $S := \{t_1, t_2, \ldots, t_{|S|}\} \subseteq T$ as the set of times $t$ at which any report is made\(^\text{10}\)

$$S := \{t \in T \mid r^\gamma(m,t) > 0 \text{ for some } \gamma, m\} \quad (5)$$

**Proposition 1.** In any equilibrium, any report $(m,t)$ for $m \in \{G,B\}$ is ever made by an incompetent expert if and only if it is ever made by a competent expert: $r^C(m,t) > 0$ if and only if $r^I(m,t) > 0$.

The reasoning behind this proposition is as follows. Suppose that there exists $(m,t)$ such that $r^I(m,t) > 0$ but $r^C(m,t) = 0$, i.e., report $m$ at $t$ is only ever made by a quack. Then after report $(m,t)$ the observer infers that the expert is surely incompetent. This renders report $(m,t)$ to be a dominated reporting strategy for the expert – strictly so if we recall that belief the expert’s type is a martingale. Therefore, there must exist another continuation strategy at time $t$ – i.e., at public history $h^p_t = (t, \emptyset)$ – that results in strictly positive reputation for at least one period. No expert is willing to play strictly dominated strategies, hence this cannot happen in equilibrium. A similar logic is in play in the opposite case – if $r^C(m,t) > 0$ and $r^I(m,t) = 0$ for some $(m,t)$ – except then reporting $(m,t)$ is a strictly dominant strategy for any type of the expert since it yields the maximal possible reputation starting from $t$ for the rest of the game. Reporting $(m,t)$ is then strictly preferred by the quack to any other alternative, which again gives a contradiction\(^\text{11}\).

4.4 Informative Reports and Babbling

If report $(m,t)$ is made in equilibrium, this does not by itself mean that it contains any meaningful information about state of the world or the type of the expert. Following Crawford and Sobel \citeyear{1982}, we refer to uninformative reports as babbling.

**Definition.** We say that report $(m,t)$ is babbling if

$$b(m,t) = b_{t-1} \quad (6)$$

$$p(m,t) = p_{t-1} \quad (7)$$

Report $(m,t)$ is informative if it is not babbling.

Condition (6) implies that the report is uninformative about the expert’s type, while (7) implies that it contains no information about the state. These two conditions can be written in terms of informativeness measures as $g(m,t) = 0$ and $i(m,t) = 0$ respectively.

It turns out that due to restriction that an expert can send at most one report, babbling reports in any equilibrium are organized in a specific structure. This is illustrated by the next proposition.

\(^{10}\)More generally, support $S$ is a subset of public histories $h^p_t$ for which $r^\gamma_t(m,t) > 0$ for some $\gamma, \eta, m$. Since a public history in our model consists of current time $t$ and a messaging history $\mu_t$, and reports can only be made at histories with $\mu_t = \emptyset$, it is enough to define the support as a set of times.

\(^{11}\)One may easily show using the same kind of argument that $r^C(G,t) + r^C(B,t) < 1$ if and only if $r^I(G,t) + r^I(B,t) < 1$. I.e., a quack stays silent up until time $t$ with positive probability if and only if so does maven. See the proof of Proposition \(^\text{11}\) for details.
Proposition 2 (Babbling Property). Every equilibrium contains a Godwin point \( \bar{t} := \min\{t \in T | V_{t,\emptyset}^C = V_{t}^I\} \) such that:

1. All on-path reports \((m, t)\) with \( t > \bar{t}\) are babbling.

2. No on-path reports \((m, t)\) with \( t \leq \bar{t}\) are babbling. Moreover:
   - at every \( t < \bar{t} \) the competent expert makes a report only if he has received the corresponding signal, i.e., \( r_{C}^{C}(m, t) = 0 \) and \( r_{\eta}^{C}(m, t) = 0 \) whenever \( \eta \neq m \);
   - at \( t = \bar{t} \) the informed competent expert always reports his signal, i.e., \( r_{\eta}^{C}(m, \bar{t}) = 1 \) for \( \eta = m \).

“Godwin’s law” states that as a discussion on the Internet continues for long enough, the probability of a comparison involving Nazis or Hitler approaches \( \frac{1}{12} \). At that point the informative part of the discussion is usually considered finished, and what follows is just babbling. Along similar lines, Proposition 2 says that in our model all equilibria feature at most two phases: early reports are informative, while the late ones do not contain any relevant information about the state or the type of the expert.

To understand Proposition 2 it is enough to note that by Proposition 1, \( t \in S \) only if a competent expert is willing to report at \( t \). His comparative advantage relative to quack is his ability to acquire private signals. Therefore, maven is only willing to participate in babbling if he has no option to exploit his [current or possibly future] information by sending an informative report – i.e., if the Godwin point \( \bar{t} \) has passed and the discourse has descended into babbling. Conversely, whenever an option to make an informative report now or in the future is present (i.e., \( t < \bar{t} \)), maven is not willing to report contrary to his private information or make an unfounded report. The only kind of information distortion that he is willing to partake in is delaying information revelation – but even in this case delaying beyond the Godwin point \( \bar{t} \) cannot be worth it.

Of course, \( \bar{t} \) does not have to be in the interior of the support, so one of the phases may be absent. In particular, if \( \bar{t} < t_1 \) then all reports are babbling, while if \( \bar{t} = t_{|S|} \) then no babbling takes place in equilibrium. We shall refer to the latter type of equilibria as informative.

Definition. An informative equilibrium is an equilibrium where all reports in the support are informative.

Note that in any informative equilibrium with support \( S \), it must be that \( \bar{t} = t_{|S|} \), since the definition directly implies \( \bar{t} \geq t_{|S|} \), and the condition \( V_{t,\emptyset}^C = V_{t}^I \) is satisfied for \( t = t_{|S|} \).

The next corollary shows that the babbling phase may be safely ignored altogether, and without loss of generality we may consider only informative equilibria.

Corollary 3 (Babbling Irrelevance). For any equilibrium with support \( S \) and Godwin point \( \bar{t} \) there exists an informative equilibrium with the same Godwin point \( \bar{t} \) and support \( \hat{S} = S \cap \{t \leq \bar{t}\} \) such that the two equilibria are:

1. payoff-equivalent for all players,

2. strategy-equivalent on \( \hat{S} \).

\(^{12}\)See “Meme, Counter-Meme” (Wired)
Proposition 2 and Corollary 3 together imply that any equilibrium strategy profile with some Godwin point \( \vec{t} \) can be obtained from a respective informative equilibrium with the same Godwin point by allowing for some babbling in \( \{\vec{t} + 1, ..., T\} \).

4.5 Informative Equilibria

This section fixes an arbitrary support \( S = \{t_1, t_2, ..., t_{|S|-1}, t_{|S|} = \vec{t}\} \subseteq T \) and explores properties of informative equilibria on \( S \) (assuming they exist). For simplicity we also assume throughout the remainder of Section 4 that maven’s signals are absolutely precise (\( \rho = 1 \)); this assumption is relaxed in Section 5.3.

To start with, it is useful to understand how equilibria look conditional on the support. Proposition 2 and Corollary 3 imply that the competent expert only reports when he has already received a private signal, except maybe the last point of the support. Assumption (SY) then implies that if \( r^C(G, t) > 0 \) then \( r^C(B, t) > 0 \) and vice versa. Proposition 1 together with the above leads to the fact that in any informative equilibrium for any \( t \in S \): (1) both reports \( m = G \) and \( m = B \) are made at \( t \) in equilibrium, and (2) both types of experts make any given report \( m \in \{G, B\} \) at \( t \) in equilibrium. Alternatively, one may say that \( S = \{t \mid r^C_\eta(\eta, t) > 0\} \) for any \( \eta \), i.e., in any informative equilibrium the support is a set of times at which maven discloses some of the information he possesses.

Presented next is the central result of our paper, which describes the informational content of reports and the informativeness dynamics. All monotonicity statements in this Theorem are understood in the sense of weak monotonicity.

Theorem 1. Suppose that \( |S| \geq 3 \) and an equilibrium on \( S \) exists. Then in any such equilibrium the following are true for both \( m \in \{G, B\} \):

1. later reports are less informative about state: \( |i(m, t)| \) is a decreasing function of \( t \) on \( S \);
2. reputation of a silent expert improves over time: \( b_t \) is increasing in \( t \) on \( S \setminus \{\vec{t}\} \) and constant on \( T \setminus S \);
3. making any report (with at most one exception) involves an immediate hit to reputation: \( b(m, t) \leq b_t \) for any \( t \in S \setminus \{\vec{t}\} \).

Theorem 1 starts by stating that in any informative equilibrium with \( |S| \geq 3 \) reports should become [weakly] noisier over time. This is required to provide incentives for the competent expert to disclose the information he possesses. To elaborate, Proposition 1 implies that an incompetent expert should be indifferent between all reports \( (m, t) \) made in equilibrium. At the same time, the only difference between the maven’s and the quack’s payoffs comes from their respective probabilities of guessing the state correctly with their report. Therefore, conditional on the quack’s indifference, maven with information \( \eta \in \{G, B\} \) in period \( t \) effectively maximizes the net premium for guessing the state correctly, as given by

\[
\Delta w_\eta(m, \tau) := w \left( \frac{b^\eta(m, \tau)}{1 - b^\eta(m, \tau)} \right) - w \left( \frac{b^{-\eta}(m, \tau)}{1 - b^{-\eta}(m, \tau)} \right),
\]

The statement is true for all \( t \in S \setminus \{t_1, \vec{t}\} \), and there may exist at most one \( m \in \{G, B\} \) such that \( b(m, t_1) > b_t \).
over all reports \((m, \tau)\) with \(\tau \geq t\). From (ML) and (SY) we know that \(\Delta w_\eta(m, \tau)\) is weakly positive for \(m = \eta\) and is weakly negative for \(m = -\eta\), hence it is enough to consider \(m = \eta\). Moreover, Propositions 1 and 2 together imply that in informative equilibria, \(t \in S\) if and only if an informed maven reports at \(t\). This means that for \(t \in S\) we have

\[
(\eta, t) = \max_{m, \tau \in S, \tau \geq t} \Delta w_\eta(m, \tau)
\]

or, simply speaking, \(\Delta w_\eta(m, t)\) must be a weakly decreasing function of \(t\) on \(S\) for \(m = \eta\). Note that given \(\rho = 1\), Proposition 2 implies that \(b^{-\eta}(\eta, t) = 0\) for all \(t \in S\}\{\bar{t}\}, and therefore \(\Delta w_\eta(\eta, t) = w\left(\frac{\eta'^{(\eta, t)}}{1-\eta'^{(\eta, t)}}\right)\). Finally, as \(w(x)\) is strictly increasing, its monotonicity is equivalent to monotonicity of \(b^{\eta}(\eta, t)\), which in the end directly translates into that of \(\eta(\eta, t)\). At \(t = \bar{t}\) we can have \(b^{-\eta}(\eta, t) > 0\), but given that it is due to uninformative reports made by the uninformed maven, the informativeness can only decrease further in this case.

The second and the third statements can also be easily shown using the monotonicity of \(b^{\eta}(m, t)\) derived above, but the main intuition behind them comes from quack’s indifference between all reports made in equilibrium. Take some \(t_k \in S\) and suppose that \(b(m, t_k) < b_k\) for both \(m \in \{G, B\}\). Then it should be that \(b(m, t_{k+1}) < b(m, t_k)\) for both \(m\), since otherwise any report \((m, t_k)\) dominates any report \((m, t_{k+1})\) – the former grants higher payoff at \(t_k\), higher payoff between \(t_{k+1}\) and \(T\), and higher continuation payoff after \(T\). By martingale property of beliefs, \(b(m, t_{k+1})\) and \(b_{t_{k+1}}\) should average out to \(b_k\), so in the end we have that

\[
b(m, t_{k+1}) < b(m, t_k) < b_k < b_{t_{k+1}}
\]

whenever \(b(m, t_k) < b_k\). The same argument extends to all \(t \in S\), granting the second and third statements of Theorem 1.\footnote{The two lines are staggered for illustrativeness; formally they coincide for \(t < 3\).} This argument does not preclude monotonicity from going the other way if we start from the inequality \(b(m, t_k) > b_k\) – but this case would generate a sequence \(b^{\eta}(m, t)\) that is increasing in \(t \in S\), which is incompatible with maven’s preferences discussed previously. Finally, the argument above implies that penalties for reporting increase over time: if \(t_k, t_{k+1}, t_{k+2} \in S\) then \(b_{t_{k+1}} - b(m, t_{k+2}) > b_k - b(m, t_{k+1})\). This is exemplified in Figure 2 where the red solid line shows reputation path of an expert who makes a report at \(t = 3\), and the blue dashed line shows that for \(t = 4\).\footnote{The two lines are staggered for illustrativeness; formally they coincide for \(t < 3\).}

Theorem 1 assumes \(|S| \geq 3\). Its first two statements in each case are about dynamics, so they only make sense if \(|S| \geq 2\). The reason two points in the support are not enough for us to make a sharp prediction is that the Godwin point \(\bar{t}\) is special. Its distinctive feature is allowing \(r^{\eta}_{\langle G, B\rangle}(m, \bar{t}) > 0\) – that an uninformed maven makes a report, – while from Proposition 2 we know that \(r^{\eta}_{\langle G, B\rangle}(m, t) = 0\) for all \(t < \bar{t}\). This can generate situations in which statements 2 and 3 are no longer true at \(\bar{t}\), i.e., some report \(m\) may have \(b(m, \bar{t}) > b_t\), while silence would decrease \(b_t\).

Finally, it is worth noting that even the third statement, which is inherently static, requires \(|S| \geq 2\). If \(|S| = 1\) (so \(S = \{\bar{t}\}\)) then it is no longer true: one may construct an equilibrium with \(b(m, \bar{t}) > b_\tau\) for both \(m \in \{G, B\}\). In such equilibrium either report is more likely to be made by a maven than a quack. “Static” equilibria (those with \(|S| = 1\) are in this sense potentially more informative than “dynamic” equilibria, and allowing for reports to be made at more than one point
in time may actually be harmful to the informativeness of these reports. This issue is investigated more closely in Sections 4.6 and 4.7.

4.6 Existence of Informative Equilibria

So far we have discussed properties of equilibria without proving that any equilibria actually exist, but existence of informative equilibria is not a trivial concern. The following Proposition outlines some necessary and sufficient conditions for existence of informative equilibria, which allow to understand some driving forces behind their (non)existence.

Proposition 4. Suppose $w(x)$ is a continuous function. Then

1. For any $\bar{t}$ there exists an informative equilibrium with $S = \{\bar{t}\}$;

2. If $w(x)$ is convex and $p_0 = \frac{1}{2}$ then an informative equilibrium with arbitrary $S$ exists;

3. If $w(x) = x^\alpha$ and $\alpha < 1$ then no informative equilibrium with $|S| \geq 3$ exists;

Part 1 of Proposition 4 states that at least some informative equilibria always exist. In particular, there always exist equilibria with singleton support, whatever the single period in the support is. At this period competent experts reveal all private information they have accumulated by then, and any expert without private information is also free to make a report in the hopes of guessing the state correctly.

However, the main focus of this paper is on the dynamics of announcements, so we are particularly interested in equilibria with $|S| > 1$. Part 2 of Proposition 4 gives a sufficient condition for their existence, which is convexity of the payoff function $w(x)$ and symmetry of the two states, $p_0 = \frac{1}{2}$. The model features enough continuity, so the condition on $p_0$ can be relaxed to some extent. All else equal, for any convex $w(x)$ there exists $\varepsilon > 0$ such that an informative equilibrium for arbitrary support $S$ exists whenever $p_0 \in \left(\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon\right)$.

Necessary conditions, on the other hand, are not easily obtainable in our model. The reason lies in the fact that only finite number of payoffs are used in any equilibrium. In particular, given some payoff function $w(x)$ and some equilibrium of the game, we can change values that $w(x)$ takes

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15Babbling equilibria, on the other hand, always trivially exist.
almost everywhere without affecting the equilibrium. This makes necessary conditions difficult to formulate without restricting payoff functions to a specific class, which is what the last part of Proposition 4 does. It states that for at least some class of concave payoff functions the existence of equilibria with large supports ($|S| \geq 3$) completely breaks down. Parts 1 and 3 together illustrate that main hurdles to existence are tied to intertemporal choice: if the expert has no choice of when to make a report then existence is certain, while allowing predictions to be made at multiple points in time may in some settings lead to complete breakdown of communication.

The reason for non-existence is connected to maven’s dynamic incentive compatibility constraints. This is because $t \in S$ if and only if informed maven makes a report at $t$, so he should be willing to do so instead of delaying his report until a later date. This leads to phenomena described in parts 2 and 3 of Theorem 1. In particular, any report has to drop expert’s reputation, so the only reason to make the report for the quack is a gamble for the terminal reputation: he should be willing enough to make a guess understanding it may be incorrect. A certain degree of risk-loving on behalf of the expert is required for such strategy profile to constitute an equilibrium. Conversely, if quack is too risk-averse then strategy profiles with $|S| \geq 3$ cannot satisfy incentives of both expert types at the same time. The formal argument is somewhat more subtle and can be found in the Appendix.

The intuition above naturally leads to the question: do there exist equilibria, given enough risk-aversion on expert’s behalf, in which the quack is too reluctant to make his report for the fear of guessing it wrong? Such equilibria do not require sustaining quack’s indifference between all reports, so they should seemingly exist under wider range of parameters and functional forms. In the current setting the existence of such equilibria violates Proposition 4 and is therefore impossible. In Section 5.2 we show that after adopting an alternative assumption on off-path beliefs such equilibria can in fact exist, but only if $\rho = 1$.

4.7 Comparison of Equilibria

In this section we study how informativeness of the reports depends on the shape of equilibrium. Simply speaking, we are trying to answer the question of which equilibria are more informative.

We have two characteristics that describe how informative a given equilibrium is: its support $S$ and two functions $i(m, t)$ for $m \in \{G, B\}$. Their exact meaning, however, is worth clarifying. Informativeness measure $|i(m, t)|$ is effectively a signal-to-noise ratio: it shows how noisy a given message is, conditional on the event that this message is sent. The probability of the latter, however, is governed by $S$, so $|i(m, t)|$ alone does not allow to conclude ex ante how much information will be conveyed at $t$. Sparser support $S$ means that reports arrive more rarely in equilibrium and it may take longer for a given piece of information to be disclosed, but it does not necessarily imply that less information is transmitted (as long as $\bar{t}$ is unaffected). To elaborate, any piece of information that is observed by maven at $t' \notin S$ is not lost to the void – its revelation is delayed.

16The jump from $|S| = 1$ to $|S| \geq 3$ is again tied to the special features of the Godwin point, which precludes us from making sharp statements about equilibria with $|S| = 2$.

17Remember that $w(x)$ is a function of $b_t$, which itself is a convex function of $b_t$. Therefore, even with $\alpha = 1$ the expert is still risk-loving, so all talks of risk-loving and risk-aversion are in the relative sense (one may easily verify that coefficients of both absolute and relative risk-aversion are monotone in $\alpha$).

18This discussion implicitly focuses on the observer’s welfare. Expert’s type is not of interest to the observer, hence $g(m, t)$ is not a variable of interest for us.
until \( t'' = \min\{t \in S | t > t'\} \) but it is reported eventually.

Proposition 5 below summarizes our knowledge of how different equilibria of the game compare to each other in terms of informativeness, given some fixed underlying fundamentals.

**Proposition 5.** Assume that two informative equilibria exist with respective strategy profiles \( \{r_i^0(m,t)\} \) and \( \{\bar{r}_i^0(m,t)\} \), supports \( S \) and \( \bar{S} \), and informativeness measures \( i(m,t) \) and \( \bar{i}(m,t) \). Then

1. If \( S = \{t_1, \ldots, t_k\} \), \( \bar{S} = \{t_1, \ldots, t_k, t_{k+1}, \ldots, t_{k+n}\} \) for some \( k, n \) and \( r_i^C(m,t_k) = \bar{r}_i^C(m,t_{k+n}) = 0 \) for both \( m \in \{G, B\} \) then \( |i(m,t)| \leq |\bar{i}(m,t)| \) for all \( m \in \{G, B\} \) and all \( \bar{t} \in \bar{S} \).

2. Suppose \( w(x) = x^\alpha \) for some \( \alpha > 0 \). If \( S = \bar{S} \), \( r_i^C(m,\bar{t}) < \bar{r}_i^C(m,\bar{t}) \) and \( r_i^C(-m,\bar{t}) = \bar{r}_i^C(-m,\bar{t}) \) for some \( m \in \{G, B\} \) then for all \( t \in S \setminus \{\bar{t}\} \) and all \( m \): \( |i(m,t)| > |\bar{i}(m,t)| \) if \( \alpha > 1 \) and \( |i(m,t)| < |\bar{i}(m,t)| \) if \( \alpha < 1 \).

The first part of the proposition says that expanding support to the right increases the informativeness of all reports as long as maven makes no uninformed reports. In other words, this says that “extending the deadline” for reports is always good: it both allows more information to be transmitted by the informed maven (in case he observes his private information between \( t_k \) and \( t_{k+n} \)) and decreases noise of all informative reports (weakly for all \( t \leq t_k \) and strictly for all \( t > t_k \)). The intuition behind the latter phenomenon follows from Theorem 1. Simply speaking, the more reporting options are available to quack, the thinner he spreads over them. A more detailed argument follows.

Ceteris paribus, extending support to the right (i.e., adding later dates) implies that reputation \( b_t \) of the silent expert should improve at the new dates. This makes silence more attractive to quack and does not affect his payoff from making a report. By Proposition 4 quack should be indifferent between staying silent and making any report, so to restore this indifference after expanding support we have to make reports more appealing to him – which is achieved by prescribing pointwise lower \( r^C(m,t) \) in equilibrium, thereby improving \( b(m,t) \) and \( b^C(m,t) \) and at the same time depressing \( b_t \).

The second part of Proposition 5 explores the effect of maven’s uninformed reports, which by Theorem 1 can only take place at the Godwin point \( \bar{t} \). The statement shows that they have general equilibrium effects that, potentially surprisingly, depend on \( w(x) \). If \( w(x) \) is convex \( (\alpha > 1) \) then the effect is expected – increasing \( r_i^C(-m,\bar{t}) \) lowers the informativeness of report \( (-m,\bar{t}) \) and hence leads to lower informativeness \( |i(m,t)| \) for \( t \in S \setminus \{\bar{t}\} \) to maintain quack’s indifference between all reports.

The intuition for \( \alpha < 1 \) is more complex. Recall that increasing \( r_i^C(-m,\bar{t}) \) does not simply lower \( |i(-m,\bar{t})| \), which is equivalent to lowering \( b(-m,\bar{t}) \) and \( b^C(-m,\bar{t}) \), but also raises \( b^m(-m,\bar{t}) \) away from zero. Due to concavity of \( w(x) \), the latter effect dominates the former in terms of payoff, so increasing \( r_i^C(-m,\bar{t}) \) actually makes report \( (-m,\bar{t}) \) more appealing to the incompetent expert. This means that \( |i(m,t)| \) for other reports should increase as well to maintain his indifference.\(^{19}\)

\(^{19}\text{Albeit recall that with } \alpha < 1 \text{ there may be at most two dates in the support by part 3 of Proposition 4.}\)
5 Discussion and Extensions

5.1 Delay Equilibria

Although Proposition 2 states that a competent expert only reports at \( t < \bar{t} \) if he has already received a signal, it is still possible that he may delay his report, making it some time after he has received a signal (but no later than \( \bar{t} \)). If this happens, we call an equilibrium a delay equilibrium. Conversely, if a competent expert always discloses his information immediately then we call it a relay equilibrium.

**Definition.** We call an informative equilibrium:

1. a *relay equilibrium* if \( r_C^T(\eta, t) = 1 \) for \( \eta \in \{G, B\} \) and for all \( t \in S \);
2. a *delay equilibrium* otherwise.

Delay equilibria are very special in two respects. Firstly, unlike relay equilibria, they only exist under knife-edge conditions on parameters. In other words, a generic informative equilibrium is a relay equilibrium, in which the competent expert discloses his signals immediately. Secondly, delay equilibria necessarily possess more concrete properties than relay equilibria. In particular, Proposition 6 describes how equilibrium properties described in Theorem 1 specialize in case of delay equilibria.

**Proposition 6.** Suppose that \( |S| \geq 3 \) and a delay equilibrium on \( S \) exists. Then in any such equilibrium the following are true for both \( m \in \{G, B\} \):

1. report informativeness \( |i(m, t)| \) is constant for all \( t \in S \setminus \{\bar{t}\} \);
2. silent expert’s reputation is independent of time: \( b_t \) is constant on \( T \setminus \{\bar{t}\} \);
3. expert’s reputation is not immediately affected by his report: \( b(m, t) = b_t \) for any \( t \in S \setminus \{\bar{t}\} \).

Both observations above (that existence conditions and equilibrium properties of delay equilibria present a special case of those for relay equilibria) stem from a common source. In comparison to relay equilibria, delay equilibria impose an extra set of restrictions on players’ payoffs: the informed competent expert must be indifferent between revealing his signal today and delaying his report until the next \( t \in S \). Given that this should be satisfied for both kinds of private signals together with quack’s indifference, the set of compatible equilibrium belief profiles shrinks significantly which allows us to provide a significantly stronger version of Theorem 1 for delay equilibria.

5.2 Ideal Equilibria

Informative equilibria with nontrivial supports need not exist with non-convex payoffs, as evidenced by Proposition 4. A question arises: are babbling and small-support equilibria the only possible outcomes when experts are too risk-averse? The answer is “not necessarily”.

The key to answering this question is assumption (OP). It requires that once an expert has gained perfect reputation it persists forever – even if an expert’s prediction turned out to be wrong when it could not happen in equilibrium (which is the case if maven is supposed to report in equilibrium
only if he has the respective signal). This is a limiting case of the model as \( \rho \to 1 \), i.e., it can be supported by a perturbation of the model in which the maven’s signal is incorrect with vanishing probability – and thus so are his predictions (see Section 5.3 for a more extensive discussion of this setting).

However, this is not the only possible perturbation of the model in case \( \rho = 1 \). One may alternatively think of a version of the model with infinitesimal number of “crazy” experts who are not strategic in their reports and just voice their opinions at random times. Since their number is infinitesimal, Bayes’ rule still prescribes that \( b(m,t) = 1 \) for any \((m,t)\) such that \( r_C^\eta(m,t) > 0 = r_I(m,t) = r_C^\emptyset(m,t) \) with \( \eta = m \). However, since an informed competent expert is never wrong, if such prediction \((m,t)\) turns out incorrect, this would imply that it was actually made by one of the few crazy experts who may be competent or not. This could lead to any belief \( b^{-m}(m,t) \in [0,1] \).

In this section we substitute (OP) by an alternative assumption (OP’) which prescribes the worst possible off-path belief after an incorrect prediction supposedly made by a competent expert, same as any other off-path history:

(\text{OP}) off the equilibrium path the beliefs are \( p = p_0 \) and \( b = 0 \), with the exception that extreme belief \( b = 1 \) is not updated;

(\text{OP’}) off the equilibrium path the beliefs are \( p = p_0 \) and \( b = 0 \).

The alternative assumption (OP’) allows for the existence of ideal equilibria:

\textbf{Definition.} Ideal equilibria are characterized by \( r_I(m,t) = r_C^\emptyset(m,t) = 0 \) for all \((m,t)\), \( r_C^\eta(m,t) = 0 \) for \( \eta \neq m \), and \( r_C^\eta(m,t) > 0 \) for some \((m,t)\) with \( m = \eta \).

Simply speaking, in ideal equilibria the only reports that are ever made are those by informed mavens; quacks never voice their opinion. This type of equilibrium is enforced by worst possible terminal reputation if the expert’s report turned out incorrect. For this threat to enforce such equilibrium, the incompetent experts should be afraid of bad reputation more than they should love good reputation in the short term. This condition is formalized by the following proposition.

\textbf{Proposition 7.} Under (OP’) ideal equilibrium with support \( S \) exists if and only if

\[
\lim_{x \to +\infty} w(x) \leq \sum_{k=1}^{\vert S \vert} \frac{t_{k+1} - t_k}{T - t_1 + p_0 \theta} \cdot w \left( \frac{b_0}{1 - b_0} (1 - F(t_k)) \right) + \frac{T - t + \theta}{T - t_1 + p_0 \theta} \cdot w \left( \frac{b_0}{1 - b_0} (1 - F(t)) \right).
\]

The equation in Proposition 7 is the analog of equation (24) in the proof of Proposition 4. It can be reduced to the following sufficient condition:

\[
(T - t_1 + p_0 \theta) \cdot \lim_{x \to +\infty} w(x) \leq (T - t_1 + \theta) \cdot w \left( \frac{b_0}{1 - b_0} (1 - F(t)) \right).
\]

Either condition above requires that \( w(x) \) is bounded (i.e., that \( \lim_{x \to +\infty} w(x) \) is finite), which means that no ideal equilibria exist if \( w(x) \) is convex. This is intuitive: in the absence of private information the expert should be sufficiently reluctant to make a report and risk losing his reputation if the report turns out incorrect – even if he is deemed absolutely competent otherwise. This is the
exact opposite of part 1 of Proposition 4 meaning that ideal equilibria are, informally speaking, complementary to informative equilibria in the sense of existence.

5.3 Imperfect Private Signals

This section provides the long-anticipated answer to the question “what if maven’s information is noisy ($\rho < 1$)?”. Until now we have assumed that maven’s signal is perfectly informative about the state. However, there is nothing in the intuition behind Theorem 1 implying that this is a necessary condition. As long as $\rho > \frac{1}{2}$, maven’s signal is somewhat informative about the state, so his informed report about the state is more likely to be supported by the ex post evidence than quack’s random guess. Therefore, the results should continue to hold.

The proofs of Propositions 1, 2, and Corollary 3 continue to hold in case $\rho < 1$ with no further modifications. Proposition 8 below shows that the remaining results continue to hold as well if $w(x)$ is either convex, or at least not too globally concave and private signal is sufficiently precise.20

Proposition 8. Theorem 1, Propositions 4–6 are true for $\rho < 1$ if either of the following holds:

- $w(x)$ is convex;
- $w(x)$ is continuously differentiable and there exist $0 < d \leq \bar{d} < +\infty$ such that $w'(x) \in [d, \bar{d}]$ and $\rho > \frac{d}{\bar{d} + d}$.

When describing the intuition behind Theorem 1, we have mentioned that in order to provide incentives for the informed maven to reveal his private information immediately instead of waiting for a later date, the premium $\Delta w_\eta(m, t)$ for guessing the state correctly should be a decreasing function of $t$ on $S$. An important part of the proof of Theorem 1 consists of showing that decreasing $\Delta w_\eta(m, t)$ is equivalent to decreasing $b^m(m, t)$. The three statements of Theorem 1 then follow almost directly from the latter statement (using the Bayes’ rule and the martingale property of beliefs).

The equivalence relation above is simple when $\rho = 1$, since then $b^-m(m, t) = 0$, and $w(x)$ is a strictly increasing function. Proposition 8 provides two alternative conditions under which the equivalence holds in case $\rho < 1$. In case when $w(x)$ is convex it holds because $b^m(m, t)$ and $b^-m(m, t)$ are scalar multiples of each other.21 The second condition relaxes convexity to just bounded derivative of $w(x)$ but the idea is the same: if $w'(x)$ is bounded so that $w(x)$ is not too concave globally, and the signal is precise enough, we can establish the connection between $\Delta w_\eta(m, t)$ and $b^m(m, t)$.

It is also worth noting that ideal equilibria outlined in Section 5.2 can no longer exist if $\rho \in (\frac{1}{2}, 1)$. This is because the expert who is believed competent with probability one can no longer be punished after his prediction was revealed to be wrong – he can credibly claim that the mistake was made because of an incorrect private signal, rather than due to low competence.

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20 The exception is Proposition 7 since assumption (OP’) is equivalent to (OP) when $\rho < 1$.

21 This follows from the observer’s belief $p_t$ regarding state being constant in the absence of reports and the rate of arrival of maven’s private signal being the same in both states. Due to these assumptions, ratio of $b^m(m, t)$ to $b^-m(m, t)$ equals the relative probability of maven having correct versus incorrect information about the state.
6 Conclusion

The paper presents a model of dynamic cheap talk in the presence of career concerns. We discover that dynamic incentive compatibility constraints shape equilibria in a particular way. Competent experts must be incentivized to at least partially reveal their information as soon as they get it, and this can be implemented in equilibrium only through more noise in later predictions. At the same time, to restrict quacks’ desire to make unfounded predictions it should be the case that making a report hurts the expert’s reputation in the short-run, and later predictions hurt it more than earlier ones.

Apart from reports’ informativeness dynamics and expert’s reputation dynamics, we identify conditions for existence of various types of equilibria. We show that convexity of payoffs with respect to reputation is sufficient and vaguely necessary for existence of equilibria in which reports are made at multiple dates. Finally, we make some statements about how different equilibria compare in terms of the amount of noise in the reports made in these equilibria.

The model can be extended in multiple directions, e.g., to account for competition among experts or for arrival of public signal in the background. Richer private news processes for experts can also add another strategic layer to the timing decision of expert’s prediction. All of these are prospective avenues for future research.

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**Appendix**

**Proof of Proposition 1** The proof is valid for all $\rho \in (\tfrac{1}{2}, 1]$. We show that $r^C(m, t) > 0$ if and only if $r^I(m, t) > 0$ for any $(m, t)$ for any history with $b(h^b_t) \in (0, 1)$. Together with the fact that $b_0 \in (0, 1)$, this will then mean that on equilibrium path we never arrive at a [non-terminal] history with $b(h^b_t) \in \{0, 1\}$, hence the statement is true for all histories on equilibrium path.

**Part 1:** $r^C(m, t) > 0 \Rightarrow r^I(m, t) > 0$. Suppose by contradiction that $r^I(m, t) = 0$. Then $b(m, t) = 1$, meaning that $W^I_t(m, t)$ attains maximum among all continuation payoffs (feasible or not). The initial assumption $r^I(m, t) = 0$ then means that either $W^I_t(\emptyset)$, or $W^I_t(m, s)$ for some $m$ and $s > t$ attain maximum, since one of these options should be more appealing to the incompetent expert than report $(m, t)$. These payoffs, however, cannot attain maximum, since $b_t < 1$.

**Part 2:** $r^I(m, t) > 0 \Rightarrow r^C(m, t) > 0$. Suppose that $r^C(m, t) = 0$. Since $r^I(m, t) > 0$, we have $b(m, t) = 0$, and hence $W^I_t(m, t)$ attains minimum among all continuation payoffs. However, since belief about the expert’s type is a martingale from the observer’s point of view, we have either $b(\neg m, t) > 0$ or $b_{t+1} > 0$. Thus at least one of these strategies (reporting $\neg m$ or staying silent at $t$) strictly dominates the strategy of reporting $(m, t)$ for incompetent expert, so $r^I(m, t) = 0$.

Similarly, one can show that $r^I(G, s) + r^I(B, s) = 1$ if and only if $r^C(G, s) + r^C(B, s) = 1$. Indeed, if for some $t$ we have $r^C(G, t) + r^C(B, t) = 1$ and $r^I(G, t) + r^I(B, t) < 1$, then not making a report by
t grants the incompetent expert a continuation payoff of zero, while by the martingale property of belief there exists \( m \in \{ G, B \} \) such that \( b(m, t) > 0 \), and therefore report \((m, t) \) dominates the strategy of staying silent. Similarly, if \( r^C(G, t) + r^C(B, t) < 1 \) and \( r^f(G, t) + r^f(B, t) = 1 \), then not making a report by \( t \) yields the maximal continuation payoff, while again by the martingale property making at least some report gives strictly less in expectation.

Before we proceed, it is useful to introduce some new pieces of notation which come in handy for further proofs. Competent expert’s report probabilities can be rewritten as

\[
\begin{align*}
r^C(m, t) &= \mathbb{E}_\eta [r^C_\eta(m, t)] = \frac{\hat{p}_0 \cdot z_{t,G} \cdot r^C_{G}(m, t) + (1 - \hat{p}_0) \cdot z_{t,B} \cdot r^C_{B}(m, t) + z_{t,\emptyset} \cdot r^C_{\emptyset}(m, t)}{\hat{p}_0 \cdot z_{t,G} + (1 - \hat{p}_0) \cdot z_{t,B} + z_{t,\emptyset}}, \\
\mathbb{E}_\eta [r^C_\eta(m, t)|\omega] &= \frac{\rho \cdot z_{t,\omega} \cdot r^C_\omega(m, t) + (1 - \rho) \cdot z_{t,\neg\omega} \cdot r^C_{\neg\omega}(m, t) + z_{t,\emptyset} \cdot r^C_{\emptyset}(m, t)}{\rho \cdot z_{t,\omega} + (1 - \rho) \cdot z_{t,\neg\omega} + z_{t,\emptyset}},
\end{align*}
\]

where \( \hat{p}_0 = p_0 \rho + (1 - p_0)(1 - \rho) \), and \( z_{t,\eta} = P \{ \eta_t = \eta, \mu_{t-1} = \emptyset | \eta^* = \eta \} \) for \( \eta \in \{ \emptyset, G, B \} \). I.e., \( z_{t,\eta} \) is the probability that competent expert has information \( \eta \) at time \( t \) and has not made a report prior to \( t \), conditional on expert’s signal realization being \( \eta^* = \eta \) (or unconditional if \( \eta = \emptyset \)). It can be expressed recursively as

\[
\begin{align*}
z_{t,\eta} &= z_{t-1,\eta} \cdot \left( 1 - \sum_m r^C(m, t-1) \right) + z_{t-1,\emptyset} \cdot \lambda(t) \cdot \left( 1 - \sum_m r^C_{\emptyset}(m, t-1) \right), \\
z_{t,\emptyset} &= z_{t-1,\emptyset} \cdot (1 - \lambda(t)) \cdot \left( 1 - \sum_m r^C_{\emptyset}(m, t-1) \right), \tag{8}
\end{align*}
\]

with \( z_{0,G} = z_{0,B} = 0 \) and \( z_{0,\emptyset} = 1 \). In any symmetric equilibrium we have \( z_{t,G} = z_{t,B} \equiv z_t \), so the expectations above transform into

\[
\begin{align*}
r^C(m, t) &= \mathbb{E}_\eta [r^C_\eta(m, t)] = \tilde{z}_t (\hat{p}_0 r^C_{G}(m, t) + (1 - \hat{p}_0) r^C_{B}(m, t)) + (1 - \tilde{z}_t) r^C_{\emptyset}(m, t), \\
\mathbb{E}_\eta [r^C_\eta(m, t)|\omega] &= \tilde{z}_t (\rho r^C_\omega(m, t) + (1 - \rho) r^C_{\neg\omega}(m, t)) + (1 - \tilde{z}_t) r^C_{\emptyset}(m, t), \tag{9}
\end{align*}
\]

where \( \tilde{z}_t = \frac{z_t}{z_t + z_{t,\emptyset}} \) and \( 1 - \tilde{z}_t = \frac{z_{t,\emptyset}}{z_t + z_{t,\emptyset}} \).

**Proof of Proposition 2** The proof is valid for all \( \rho \in (\frac{1}{2}, 1] \). We begin with a useful observation:

\[
W^C_{t,\omega}(m, t) = W^f_t(m, t) \quad \text{for } m \in \{ G, B \}. \tag{11}
\]

If the competent expert reports \((m, t) \) before observing a private signal, his continuation payoff coincides with that of the incompetent expert, since they possess the same private information at any such history.

Note further that the existence of a Godwin point \( t = \min\{ t \in T | V^C_{t,\emptyset} = V^f_t \} \) is trivial since the required equality is always satisfied for the last point of \( S \). To see this, observe that any report \((m, t) \) for \( t > t_{|S|} \) yields zero reputation for the rest of the game due to assumption (OP), and is therefore weakly dominated for any type of the expert by staying silent. At the same time, staying silent yields the same time-\( t \) expected payoff to the uninformed (as of time \( t \)) competent expert as it does to incompetent expert, since they have the same information. This together with \ref{11} gives the result.

Most of the remaining proof is devoted to showing that \( V^C_{t,\emptyset} = V^f_t \) for some \( t \) implies babbling in all further times. This is established in a series of claims. The second Godwin point property is then easily shown by contradiction.
As a starting point, we show that \( W_{\ell,\eta}^C(m,t) = W_t^I(m,t) \) for any \( m \in \{G,B\} \), any \( \eta \in \{G,B\} \) and any \( t \in S^m \) such that \( t > \bar{t} \). Suppose the converse – there exist \( m, t \) and \( \eta \) such that \( W_{\ell,\eta}^C(m,t) \neq W_t^I(m,t) \). Then (ML) and (SY) imply that there can be three cases:

**Case 1:** \( r_G^C(G,t) = r_B^G(B,t) \geq r_G^B(G,t) = r_B^B(G,t) > 0 \).

In this case \( W_{\ell,G}^C(G,t) = W_{\ell,G}^C(B,t) \) and, by Proposition \( V_t^I = W_t^I(G,t) = W_t^I(B,t) \). Therefore, \( W_{\ell,G}^C(G,t) - W_t^I(G,t) = W_{\ell,G}^C(B,t) - W_t^I(B,t) \), which reduces to the equality of differences in terminal reputation:

\[
0 = W_t^I(G,t) - W_t^I(B,t) = \theta \cdot \frac{p_0 \cdot (1 - p_0) \cdot (2p - 1)}{\bar{p} \cdot \bar{r}} \cdot \left( w \left( \frac{b^G(m,t)}{1 - b^G(m,t)} \right) - w \left( \frac{b^B(m,t)}{1 - b^B(m,t)} \right) \right),
\]

or

\[
V_t^G = V_t^B.
\]

If \( r_G^C(G,t) > r_B^G(B,t) \) then, by (10) and the expression for \( b^G(m,t) \), the LHS of (12) is weakly positive. However, due to (SY) we then have that \( r_G^B(B,t) < r_B^C(G,t) \), so the RHS is weakly negative. The converse also holds, which leaves us with the conclusion that for (12) to be satisfied, both sides must be equal to zero. Therefore, \( b^G(m,t) = b^B(m,t) \) for any \( m \in \{G,B\} \), which implies \( W_{\ell,G}^C(m,t) = W_t^I(m,t) = V_t^I \) for any \( \eta \in \{G,B\} \) and any \( m \in \{G,B\} \), a contradiction.

**Case 2:** \( r_G^C(G,t) = r_B^G(B,t) > 0 = r_B^C(B,t) = r_G^B(G,t) \).

As \( r_G^C(G,t) > r_B^G(B,t) \) and \( r_G^B(B,t) > r_B^C(G,t) \), we have that \( V_{\ell,G}^C = W_{\ell,G}^C(G,t) > W_t^I(G,t) = V_t^I \) and, analogously, \( V_{\ell,B}^C > V_t^I \). Next, note that \( V_{\ell,\bar{s}}^C \) is, for all \( t > \bar{t} \), bounded below by

\[
\sum_{s=t}^{t-1} w \left( \frac{b_s}{1 - b_s} \right) + P(t^* \leq t \mid t^* > \bar{t}) \cdot \left( \tilde{p}_0 \cdot V_{\ell,G}^C + (1 - \tilde{p}_0) \cdot V_{\ell,B}^C \right) + P(t^* > t \mid t^* > \bar{t}) \cdot V_t^C\bar{s},
\]

which is the value of not making a report from \( \bar{t} \) until at least \( t \). By (11) we have \( V_{\ell,\bar{s}}^C \geq V_{\ell,\bar{s}}^C(m,t) = W_t^I(m,t) \). Second, we have shown that \( V_{\ell,\bar{s}}^C > V_t^I \). Therefore,

\[
V_{\ell,\bar{s}}^C \geq \sum_{s=t}^{t-1} w \left( \frac{b_s}{1 - b_s} \right) + W_t^I(m,t) = V_t^I,
\]

which gives us a contradiction with the definition of \( \bar{t} \).

**Case 3:** \( r_G^C(G,t) = r_B^G(B,t) = r_B^C(B,t) = r_G^B(G,t) = 0 \) and \( r_G^C(m,t) > 0 \) for some \( m \).

\( r_G^C(G,t) = r_B^G(B,t) = r_B^C(B,t) = r_G^B(G,t) \) automatically implies \( W_{\ell,\eta}^C(m,t) = W_t^I(m,t) \) for any \( \eta \in \{G,B\} \) and any \( m \in \{G,B\} \), which gives us a contradiction with the initial assumption.

Next we show that \( W_{\ell,\eta}^C(m,t) = W_t^I(m,t) \) for all \( \eta, m \in \{G,B\} \) with \( t \in S^m \) implies that report \( (m,t) \) is babbling. Without loss of generality assume \( \eta = G \). Expanding the equality, we see that

\[
0 = W_{\ell,G}^C(m,t) - W_t^I(m,t) = \theta \cdot \frac{p_0 \cdot (1 - p_0) \cdot (2p - 1)}{\bar{p} \cdot \bar{r}} \cdot \left( w \left( \frac{b^G(m,t)}{1 - b^G(m,t)} \right) - w \left( \frac{b^B(m,t)}{1 - b^B(m,t)} \right) \right),
\]

and therefore \( b^G(m,t) = b^B(m,t) \). It further implies that (4) reduces to (7). In other words, it follows that reputation should not be affected by the revelation of state after any time-\( t \) report.

To conclude that only babbling is possible after \( \bar{t} \) we are left to show that (6) holds for all \( (m,s) \) with \( s > \bar{t} \). Condition (3) is equivalent to \( g(m,s) = 0 \). Three cases are possible (since we have shown in the proof of Proposition that \( r^G(G,s) + r^B(B,s) = 1 \) cannot be the case for exactly one \( \gamma \)).

**Case 1:** \( s = \max \{t \in S \mid t > \bar{t}\} \) and \( r^G(G,s) + r^B(B,s) = 1 \) for any \( \gamma \in \{C,I\} \).
If \( m \) is the only report made at \( s \) then \( r^\gamma_\eta(m, s) = 1 \) for all \( \gamma, \eta \), which implies \( g(m, s) = 0 \). If both reports are made on path at \( s \), then by the same logic \( r^\gamma_\eta(G, s) + r^\gamma_\eta(B, s) = 1 \), and if \( g(m, s) \neq 0 \) for some \( m \) then the report with higher \( g(m, s) \) is strictly preferred by either expert, contradicting that both reports occur on path.

**Case 2:** \( s = \max\{t \in S \mid t > \tilde{t}\} \) and \( r^\gamma(G, s) + r^\gamma(B, s) < 1 \) for \( \gamma \in \{C, I\} \).

If \( m \) is the only report made at \( s \) and \( g(m, s) \neq 0 \) then an incompetent expert has strict preference between report \((m, s)\) and staying silent at \( s \) (because in either case he gets a degenerate lottery at \( T \), since \( \tilde{t} \) is satisfied). This strict preference cannot occur in equilibrium, thus \( g(m, s) = 0 \). If both reports are made on path at \( s \) then we can combine the two indifference arguments above to obtain that \( g(G, s) = g(B, s) \) and, consequently, \( g(m, s) = 0 \) for all \( m \in \{G, B\} \).

**Case 3:** \( s < \max\{t \in S \mid t > \tilde{t}\} \).

From the previous case we know that \( g(m, s) = 0 \) for any on-path \( m \) at \( s = \max\{t \in S \mid t > \tilde{t}\} \). We can iterate backwards from here as follows. If \( s - 1 \in S \) then an incompetent expert should be indifferent between making an on-path report at \( s - 1 \) and at \( s \), which can only happen if \( g(m, s - 1) = 0 \), because \( g(m, s) = 0 \) and \( \tilde{t} \) is satisfied for both reports. Iterating backwards we establish the claim for all \( t > \tilde{t} \). If some of these periods are not in \( S \) then they can be skipped because beliefs do not change at such periods.

All of the above proves that only babbling is possible after the Godwin point.

We are left to show the second part of the proposition. First, suppose there exist \( m \) and \( t < \tilde{t} \) such that \( r^C_G(m, t) > 0 \). Then \( V_{t,\emptyset}^C = W_{t,\emptyset}^C(m, t) = W_t^I(m, t) = V_t^I \), where the second equality follows from \([1] \). Thus \( t \geq \tilde{t} \) by definition of \( \tilde{t} \) – a contradiction.

Now suppose there exists \( t \leq \tilde{t} \) such that \( r^C_G(-\eta, t) > 0 \) for some \( \eta \in \{G, B\} \). As shown before, it implies \( V_{t,\emptyset}^C = W_{t,\emptyset}^C(m, t) = W_t^I(m, t) = V_t^I \) for all \( \eta, m \in \{G, B\} \). Suppose first that \( t = \tilde{t} \). Then as \( r^C_G(m, t_{|S| - 1}) = 0 \) for all \( m \in \{G, B\} \), we have

\[
V_{t_{|S| - 1},\emptyset}^C = w \left( \frac{b_{|S| - 1}}{1 - b_{|S| - 1}} \right) \cdot P\{t' < \tilde{t} \mid t' > t_{|S| - 1}\} \cdot \left( \bar{b}_0 \cdot V_{t_G}^C + (1 - \bar{b}_0) \cdot V_{t,B}^C \right) + P\{t' > \tilde{t} \mid t' > t_{|S| - 1}\} \cdot V_{t,\emptyset}^C.
\]

As \( V_{t,\emptyset}^C = V_t^I \) and \( V_{t,\emptyset}^C = V_t^I \), the above expression reduces to \( V_{t_{|S| - 1},\emptyset}^C = w \left( \frac{b_{|S| - 1}}{1 - b_{|S| - 1}} \right) + V_t^I = V_{t_{|S| - 1}}^I \), which constitutes a contradiction with the definition of the Godwin point. One can similarly show that \( r^C_G(\eta, \tilde{t}) > 0 \), as otherwise \( V_{t,\emptyset}^G = W_{t,\emptyset}^G(m, \tilde{t}) = W_t^I(m, \tilde{t}) = V_t^I \) for all \( \eta \in \{G, B\} \) and all \( m \in \{G, B\} \), which leads to the same contradiction with the definition of the Godwin point as above. Finally, if \( t < \tilde{t} \) then, as \( r^C_G(-\eta, \tilde{t}) = 0 \) and \( r^C_G(\eta, \tilde{t}) > 0 \) imply \( V_{t,\emptyset}^C > V_t^I \), a competent expert who has received a signal by period \( t \) can postpone his report until \( \tilde{t} \) and receive strictly more than incompetent expert which contradicts \( V_{t,\emptyset}^C = V_t^I \) implied by \( r^C_G(-\eta, \tilde{t}) > 0 \).

We are left to show that a competent expert never wants to conceal his private signal. Assume \( r^C_G(\eta, \tilde{t}) < 1 \). Then a competent expert must weakly prefer to conceal his private signal than to report it. In the first case a competent expert receives exactly \( V_t^I \), while in the latter he gets \( V_{t,\emptyset}^C > V_t^I \), – a contradiction.

**Proof of Corollary** The proof is valid for all \( \rho \in (\frac{1}{2}, 1] \). Let \( \{r^\gamma_\eta(m, t)\} \) be an equilibrium strategy profile. Consider a new strategy profile \( \{\tilde{r}^\gamma_\eta(m, t)\} \) such that \( \tilde{r}^\gamma_\eta(m, t) = r^\gamma_\eta(m, t) \), \( \tilde{r}^\gamma_I(m, t) = r^\gamma_I(m, t) \) for all \( t \leq \tilde{t} \) and \( \tilde{r}^\gamma_{t,\emptyset}(m, t) = \tilde{r}^\gamma_I(m, t) = 0 \) for all \( t > \tilde{t} \). As strategies coincide on \( \tilde{S} \) and all reports \((m, t)\) with \( t > \tilde{t} \) are babbling in the original equilibrium, the following are true:

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1. beliefs $b(m, t)$ and $b^*(m, t)$ induced by the two strategy profiles coincide for all $\omega, m \in \{G, B\}, t \in \bar{S}$;

2. belief sequences $b_t$ induced by the two strategy profiles coincide for all $t \in \bar{T}$.

The latter statement also exploits the fact that $S \setminus \bar{S}$ is nonempty (otherwise the corollary statement trivially holds), so it must be that $r^G(G, \bar{t}) + r^G(B, \bar{t}) < 1$ and $r^C(G, \bar{t}) + r^C(B, \bar{t}) < 1$.

The first statement above implies that any report $(m, t)$ with $t \leq \bar{t}$ yields the same payoff under either strategy profile. The second statement states that reporting nothing in any period yields the same payoffs as well. Strategy of reporting nothing yields the same payoff under $\{\tilde{r}^G(m, t)\}$ as any report $(m, t)$ with $t > \bar{t}$ under $\{r^G(m, t)\}$, since all such reports are babbling. Finally, any report $(m, t)$ with $t \notin S$ yields the same payoff under either strategy profile due to (OP).

Everything said above directly implies that if $r^G(m, t)$ is a best response for type-$\gamma$ expert to strategy profile $\{r^G(m, t)\}$ then $\tilde{r}^G(m, t)$ is a best response for him to strategy profile $\{\tilde{r}^G(m, t)\}$ and yields the same payoff. 

\[ \square \]

**Proof of the Main Result**

Before proceeding to the proof of Theorem 1, we provide some expressions for belief updating that will be useful in further proofs. Using Proposition 2 and the notion of $\tilde{z}_t$ introduced earlier in this Appendix, we can rewrite the expressions for (2) and (3) in a more explicit form. Proposition 2 implies that for all $t < \bar{t}$ we have $r^G_B(B, t) = r^G_B(G, t) = 0$. Therefore, (9) and (10) together imply that for all $t \in S$ we have

\[ \begin{align*}
\frac{b(G, t)}{1 - b(G, t)} &= \frac{b_{t-1}}{1 - b_{t-1}}, \quad \frac{r^G_C(G, t)}{r^G(G, t)} = \frac{b_{t-1}}{1 - b_{t-1}}, \quad \frac{\hat{p}_0\tilde{z}_t r^G_B(G, t) + (1 - \tilde{z}_t) r^G_C(G, t)}{r^G(G, t)}, \\
\frac{b(B, t)}{1 - b(B, t)} &= \frac{b_{t-1}}{1 - b_{t-1}}, \quad \frac{r^G_C(B, t)}{r^G(B, t)} = \frac{b_{t-1}}{1 - b_{t-1}}, \quad (1 - \hat{p}_0) \tilde{z}_t r^G_B(B, t) + (1 - \tilde{z}_t) r^G_C(B, t),
\end{align*} \]

and

\[ \begin{align*}
\frac{b^G(G, t)}{1 - b^G(G, t)} &= \frac{b_{t-1}}{1 - b_{t-1}}, \quad \frac{E_{\gamma}[r^G_C(G, t)|G]}{r^G(G, t)} = \frac{b_{t-1}}{1 - b_{t-1}}, \quad \frac{\rho \tilde{z}_t r^G_B(G, t) + (1 - \tilde{z}_t) r^G_C(G, t)}{r^G(B, t)}, \\
\frac{b^B(B, t)}{1 - b^B(B, t)} &= \frac{b_{t-1}}{1 - b_{t-1}}, \quad \frac{E_{\gamma}[r^G_C(B, t)|B]}{r^G(B, t)} = \frac{b_{t-1}}{1 - b_{t-1}}, \quad (1 - \rho) \tilde{z}_t r^G_B(B, t) + (1 - \tilde{z}_t) r^G_C(B, t),
\end{align*} \]

It is also worth remembering that $r^G(m, t) = 0$ for any $t < \bar{t}$.

In case no report was made in period $t < \bar{t}$, the belief is updated as

\[ \frac{b_t}{1 - b_t} = \frac{b_{t-1}}{1 - b_{t-1}}, \quad \frac{1 - r^G(G, t) - r^C(B, t)}{1 - r^G(G, t) - r^B(B, t)} = \frac{b_{t-1}}{1 - b_{t-1}}, \quad \frac{1 - \tilde{z}_t r^G_B(G, t)}{1 - r^G(G, t) - r^B(B, t)}, \]

while the analogous expression for $t = \bar{t}$ is given by

\[ \frac{b_{\bar{t}}}{1 - b_{\bar{t}}} = \frac{b_{[\bar{t}]-1}}{1 - b_{[\bar{t}]-1}}, \quad \frac{1 - r^G(G, \bar{t}) - r^C(B, \bar{t})}{1 - r^G(G, \bar{t}) - r^B(B, \bar{t})} = \frac{b_{[\bar{t}]-1}}{1 - b_{[\bar{t}]-1}}, \quad \frac{(1 - \tilde{z}_\bar{t}) \cdot (1 - r^G_B(G, \bar{t}) - r^C_B(B, \bar{t}))}{1 - r^G(G, \bar{t}) - r^B(B, \bar{t})}. \]

In (15) we use the fact that $r^G_B(G, \bar{t}) = r^B_B(B, \bar{t})$ due to (SY), and in (16) we use that $r^G_B(G, \bar{t}) = r^C_B(B, \bar{t}) = 1$ by Proposition 2.
What follows is the proof of the main result, Theorem 1. To avoid duplicating the arguments, we merge it with the proof of Proposition 8.

**Proof of Theorem 1 and Proposition 8** First, recall that Propositions 1, 2 and Corollary 3 are true for all \( \rho \in [\frac{1}{2}, 1] \), and so can be employed in this proof. Further, note that in all babbling periods \( t \) we have \( i(m, t) = 0, b(m, t) = b_t - 1 \) for \( m \in \{ G, B \} \), and \( b_t \) stays on a constant level. Together with Proposition 2 and Corollary 3, this means that it is enough to show the statement of the Theorem for informative equilibria.

We start by showing that \( \Delta w_\eta(m, t) \), which is defined as

\[
\Delta w_\eta(m, t) := w \left( \frac{b^\eta(m, t)}{1 - b^\eta(m, t)} \right) - w \left( \frac{b^{-\eta}(m, t)}{1 - b^{-\eta}(m, t)} \right),
\]

is a weakly decreasing function of \( t \) on \( S \) given \( m = \eta \) (note that \( \Delta w_G(m, t) = -\Delta w_B(m, t) \)). Suppose the competent expert has private information \( \eta = G \) at time \( t \), but has not yet made any report. He chooses a report \( (m, \tau) \) with \( \tau \geq t \) which maximizes \( W^C_{t, G}(m, \tau) \), where “making no report” is also an available option. Expanding \( W^C_{t, G}(m, \tau) \), we get the following expression:

\[
\sum_{s=1}^{\tau-1} w \left( \frac{b_s}{1 - b_s} \right) + \sum_{s=t}^{\tau-1} w \left( \frac{b(m, \tau)}{1 - b(m, \tau)} \right) \theta \left( \frac{\rho p_0}{\bar{p}_0} \right) \left( \frac{b^G(m, \tau)}{1 - b^G(m, \tau)} \right) + (1 - \rho p_0) \left( \frac{b^B(m, \tau)}{1 - b^B(m, \tau)} \right),
\]

where \( \bar{p}_0 = p_0 \rho + (1 - p_0)(1 - \rho) \).

An incompetent expert is indifferent between all such reports at time \( t \). His continuation value \( W^I_t(m, \tau) \) can similarly be written as

\[
\sum_{s=1}^{\tau-1} w \left( \frac{b_s}{1 - b_s} \right) + \sum_{s=t}^{\tau-1} w \left( \frac{b(m, \tau)}{1 - b(m, \tau)} \right) \theta \left( p_0 \right) \left( \frac{b^G(m, \tau)}{1 - b^G(m, \tau)} \right) + (1 - p_0) \left( \frac{b^B(m, \tau)}{1 - b^B(m, \tau)} \right).
\]

Given that the latter expression is constant over all \((m, \tau)\), the optimization problem of a competent expert with \( \eta = G \) becomes equivalent to maximizing the difference \( \Delta w_G(m, \tau) \) over all \( \tau \in \{S| \tau \geq t\} \) and \( m \in \{G, B\} \).

Similarly, a competent expert who has observed signal \( B \) chooses report \( (m, \tau) \) which maximizes \( \Delta w_B(m, \tau) \). Propositions 1, 2 and Corollary 3 imply that \( S = \{ t \in T \mid r^G_\eta(t, q) > 0 \} \) for any \( \eta \in \{G, B\} \), so since \( t \in S \), it must be that \((G, t)\) maximizes \( \Delta w_G(m, \tau) \) and \((B, t)\) minimizes it across all \((m, \tau)\) with \( \tau \in \{S| \tau \geq t\} \). Therefore, \( \Delta w_\eta(m, \eta) \) must be a weakly decreasing function of \( t \) on \( S \).

The second step of the proof consists in showing that for \( \eta \in \{G, B\}, \Delta w_\eta(m, t) \) is weakly decreasing on \( S \setminus \{t\} \) if and only if \( \frac{b^\eta(m, t)}{1 - b^\eta(m, t)} \) (and, consequently, \( b^\eta(m, t) \) itself) is weakly decreasing on \( S \setminus \{t\} \). We demonstrate it for all cases stated in Theorem 1 and Proposition 8 separately.

**Case 1:** \( \rho = 1 \)

This case is obvious, as then \( w \left( \frac{b^{-\eta}(m, t)}{1 - b^{-\eta}(m, t)} \right) = 0 \) for any \( \eta \in \{G, B\} \), and \( w(x) \) is a strictly increasing function.

**Case 2:** \( w(x) \) is convex and \( \rho < 1 \).

Note that since \( r^G_\eta(m, t) = 0 \) for all \( t \in S \setminus \{t\} \) from (14) we have

\[
\frac{b^{-\eta}(m, t)}{1 - b^{-\eta}(m, t)} = \frac{1 - \rho}{\rho} \cdot \frac{b^\eta(m, t)}{1 - b^\eta(m, t)},
\]

where \( \frac{1 - \rho}{\rho} \in (0, 1) \) because \( \rho > \frac{1}{2} \). Take any \( t_1 > t_2 \) with \( t_1, t_2 \in S \). Then if \( \frac{b^\eta(m, \tau_1)}{1 - b^\eta(m, \tau_1)} = x_1 > x_2 = \)
\[
\frac{b^\eta(m,\tau_1)}{1-b^\eta(m,\tau_1)} \text{ we have }
\]
\[
w(x_1) - w\left(\frac{1-\rho}{\rho}x_1\right) \geq w(x_2) - w\left(x_2 - x_1 + \frac{1-\rho}{\rho}x_1\right) > w(x_2) - w\left(\frac{1-\rho}{\rho}x_2\right),
\]
where the first inequality follows from convexity of \(w(x)\), and the second is valid because \(w(x)\) is strictly increasing.

**Case 3**: \(\rho > \frac{d}{2\pi a}\) and \(w'(x) \in [d,d]\).

Similarly to the previous case take any \(\tau_1,\tau_2 \in S\) and let \(x_1 := \frac{b^\eta(m,\tau_1)}{1-b^\eta(m,\tau_1)}, x_2 := \frac{b^\eta(m,\tau_2)}{1-b^\eta(m,\tau_2)}\). Suppose \(w(x_2) - w\left(\frac{1-\rho}{\rho}x_2\right) > w(x_1) - w\left(\frac{1-\rho}{\rho}x_1\right)\). Then
\[
0 < \left(w(x_2) - w\left(\frac{1-\rho}{\rho}x_2\right)\right) - \left(w(x_1) - w\left(\frac{1-\rho}{\rho}x_1\right)\right) < \left(\frac{\rho}{1-\rho} - \frac{\rho}{\rho}\right) \cdot \left(x_2 - x_1\right) \cdot \left(\frac{\rho}{1-\rho} - \frac{\rho}{\rho}\right).
\]
As \(\left(\frac{\rho}{1-\rho} - \frac{\rho}{\rho}\right) > 0\), we must have \(x_2 > x_1\).

Conversely, if \(x_2 > x_1\) then
\[
0 < (x_2 - x_1) \cdot \left(\frac{\rho}{1-\rho} - \frac{\rho}{\rho}\right) < \left(w(x_2) - w\left(\frac{1-\rho}{\rho}x_2\right)\right) - \left(w(x_1) - w\left(\frac{1-\rho}{\rho}x_1\right)\right),
\]
which grants the result.

We next show that whenever \(|S| \geq 3\) and an equilibrium on \(S\) exists, it must be that \(b_t \geq b(m,t_1)\) for any \(m \in \{G,B\}\). Assume there exists \(m \in \{G,B\}\) such that \(b_t < b(m,t_1)\). Incompetent expert’s value \(W^I_t(m,t_1)\) equals
\[
(T - t_1) \cdot w\left(\frac{b(m,t_1)}{1-b(m,t_1)}\right) + \theta \cdot \left(1 + p_0 \cdot w\left(\frac{b^G(m,t_1)}{1-b^G(m,t_1)}\right) + (1-p_0) \cdot w\left(\frac{b^B(m,t_1)}{1-b^B(m,t_1)}\right)\right).
\]
At the same time, \(W^I_t(m,t_2)\) is equal to
\[
(t_2 - t_1) \cdot w\left(\frac{b(m,t_2)}{1-b(m,t_2)}\right) + \theta \cdot \left(1 - p_0 \cdot w\left(\frac{b^G(m,t_2)}{1-b^G(m,t_2)}\right) + (1-p_0) \cdot w\left(\frac{b^B(m,t_2)}{1-b^B(m,t_2)}\right)\right).
\]
As \(w(x)\) is strictly increasing, and \(b_t < b(m,t_1)\), \(W^I_t(m,t_1) = W^I_t(m,t_2)\) implies
\[
(T - t_2) \cdot \left(\frac{b(m,t_1)}{1-b(m,t_1)}\right) + \theta \cdot \left(1 + p_0 \cdot w\left(\frac{b^G(m,t_1)}{1-b^G(m,t_1)}\right) + (1-p_0) \cdot w\left(\frac{b^B(m,t_1)}{1-b^B(m,t_1)}\right)\right) < \\
(T - t_2) \cdot \left(\frac{b(m,t_2)}{1-b(m,t_2)}\right) + \theta \cdot \left(1 - p_0 \cdot w\left(\frac{b^G(m,t_2)}{1-b^G(m,t_2)}\right) + (1-p_0) \cdot w\left(\frac{b^B(m,t_2)}{1-b^B(m,t_2)}\right)\right).
\]
Consequently, it must be that either \(b(m,t_1) < b(m,t_2)\), or \(b^m(m,t_1) < b^m(m,t_2)\) or \(b^{-m}(m,t_1) < b^{-m}(m,t_2)\). However, \([13]\) and \([14]\) imply that both \(b^G(m,t)\) and \(b^B(m,t)\) differ from \(b(m,t)\) by a constant factor for any \(t \in S^\prime\{t\}\) (since \(r_G^B(m,t) = 0\), so the three inequalities are equivalent. Therefore, \(b^m(m,t_1) < b^m(m,t_2)\), which contradicts \(b^m(m,t)\) being decreasing on \(S\setminus\{t\}\).

We have shown that \(b_t \geq b(m,t_1)\) for any \(m \in \{G,B\}\). Consequently, as \(b_t\) is a martingale, we have that \(b_t \geq b_0\) and \(b(m,t_1) \leq b_0\) for at least one \(m \in \{G,B\}\). As \(b(m,t_1) \leq b_t\) for \(m \in \{G,B\}\), we must have that either \(b(m,t_2) \leq b(m,t_1)\), or \(b^m(m,t_2) \leq b^m(m,t_1)\) or \(b^{-m}(m,t_2) \leq b^{-m}(m,t_1)\) to make the incompetent
expert indifferent between reports \((m, t_1)\) and \((m, t_2)\). Again, \([13]\) and \([14]\) imply that all three inequalities are equivalent, so all three have to hold. The fact that \(b_t\) is a martingale together with resulting inequalities \(b_{t_1} \geq b(m, t_1) \geq b(m, t_2)\) for \(m \in \{G, B\}\) imply \(b_{t_2} \geq b_{t_1}\). Iterating this argument further, we achieve that \(b(m, t) \leq b_t\) and \(b_t\) is increasing in \(t\) on \(S \setminus \{\bar{t}\}\).

The above proves the second and the third parts of Theorem \([1]\) \(^{22}\). It remains to show the first part. Note that, by the same inductive reasoning as above, if \(b_{t_1} > b(m, t_1)\) then \(b_{t_1} > b(m, t_{\bar{t}} - 1)\). Then it is possible to show that \(b^m(m, \bar{t}) < b^m(m, t_{\bar{t}} - 1)\). Indeed, suppose the converse. Then to make incompetent expert indifferent between reporting \(m\) at \(t = t_{\bar{t}} - 1\) and \(\bar{t}\), we must have \(b^m(m, \bar{t}) < b^m(m, t_{\bar{t}} - 1)\). But then

\[
\begin{align*}
&\left( 1 - b^m(m, t_{\bar{t}} - 1) \right) - \frac{b^m(m, t_{\bar{t}} - 1)}{1 - b^m(m, t_{\bar{t}} - 1)} < \frac{b^m(m, \bar{t})}{1 - b^m(m, \bar{t})} - \frac{b^m(m, \bar{t})}{1 - b^m(m, \bar{t})}
&\text{which contradicts the fact that } \Delta w_i(t, \tau)	ext{ is weakly decreasing in } t \text{ for } m = \eta.
\end{align*}
\]

Finally, remember that for all \(t \in S \setminus \{\overline{t}\}\) we have

\[
|i(m, t)| = \ln \left( \frac{1 + \frac{b^m(m, t)}{1 - b^m(m, \overline{t})}}{1 + \frac{b^m(m, t)}{1 - b^m(m, \overline{t})}} \right) = \ln \left( \frac{1 + \frac{b^m(m, \overline{t})}{1 - b^m(m, t)}}{1 + \frac{b^m(m, \overline{t})}{1 - b^m(m, t)}} \right),
\]

which is then a decreasing function of \(t\) on \(S \setminus \{\overline{t}\}\) as well because \(\ln (1 + x) - \ln \left( 1 + \frac{1}{\rho} x \right)\) is an increasing function of \(x\). For the last two points of \(S\) we have

\[
|i(m, t_{\bar{t}} - 1)| - |i(m, \bar{t})| = \ln \left( \frac{1 + \frac{b^m(m, t_{\bar{t}} - 1)}{1 - b^m(m, t_{\bar{t}} - 1)}}{1 + \frac{b^m(m, t_{\bar{t}} - 1)}{1 - b^m(m, t_{\bar{t}} - 1)}} \right) - \ln \left( \frac{1 + \frac{b^m(m, \bar{t})}{1 - b^m(m, \bar{t})}}{1 + \frac{b^m(m, \bar{t})}{1 - b^m(m, \bar{t})}} \right) > 0,
\]

where the last inequality follows from \(b^m(m, t_{\bar{t}} - 1) > b^m(m, \bar{t})\) and the fact that \(\rho > \frac{1}{2}\). This concludes the proof of Theorem \([1]\) for general informative equilibria. \(\square\)

We continue by presenting the proof of Proposition \([6]\) which is a special case of Theorem \([1]^{23}\) Proposition \([8]\) for delay equilibria.

**Proof of Proposition \([6]\)** Let \(\{v^m_n(m, t)\}\) constitute a delay equilibrium on \(S\). First, note that if \(b_{t_1} = b(G, t_1) = b(B, t_1)\) then \(b_t = b(m, t) = b_0\) for \(t \in S \setminus \{\overline{t}\}\). It further implies that \(b^m(m, t)\) is constant on \(S \setminus \{\overline{t}\}\). Therefore, \(i(m, t)\) is constant on \(S \setminus \{\overline{t}\}\) as well, as suggested by \((17)\), so we get all three statements.

To show that \(b_{t_1} = b(G, t_1) = b(B, t_1)\), proceed by contradiction. If there exists \(m \in \{G, B\}\) such that \(b_{t_1} > b(m, t_1)\) then \(v^m_n(m, t) = 1\) for all \(t \in S \setminus \{t_{\bar{t}} - 1\}\). Due to \((SY)\), the same applies to the other \(m\) as well. Further, if such \(m\) exists then, as shown above, \(b^m(m, \bar{t}) < b^m(m, t_{\bar{t}} - 1)\), meaning that

\[
\begin{align*}
&\left( 1 - b^m(m, t_{\bar{t}} - 1) \right) - \frac{b^m(m, t_{\bar{t}} - 1)}{1 - b^m(m, t_{\bar{t}} - 1)} > \frac{b^m(m, \bar{t})}{1 - b^m(m, \bar{t})} - \frac{b^m(m, \bar{t})}{1 - b^m(m, \bar{t})}
&\text{which follows from the fact that } w(x_1) - w\left( \frac{1 - \rho}{\rho} x_1 \right) > w(x_2) - w\left( \frac{1 - \rho}{\rho} x_2 \right) \text{ if and only if } x_1 > x_2 \text{ whenever } \rho = 1 \text{ (which corresponds to Theorem \([1]\) or any of the two conditions in Proposition \([8]\) are satisfied. Finally,}
\end{align*}
\]

\(^{22}\) The statement that \(b_t\) is constant on \(T \setminus S\) follows trivially from \([2]\).

\(^{23}\) The claim for all points except the two last ones follows from the fact that \(b^m(m, t)\) is strictly decreasing in this case. Furthermore, remember that \(v^m_n(m, t) = 1\) for \(m \in \{G, B\}\) by Proposition \([2]\).
Lemma 9. For any delay equilibrium on $S$ with $|S| \geq 3$ there exists a payoff-equivalent relay equilibrium, such that beliefs after the same histories coincide in the two equilibria.

Proof. Assume that strategy profile $\{r^G_\eta(m,t)\}$ constitutes a delay equilibrium on $S$. Consider strategy profile $\{\tilde{r}^G_\eta(m,t)\}$ such that

1. $\tilde{r}_\eta^C(m,t) = r_\eta^C(m,t)$ and $\tilde{r}_\eta^I(m,t) = r_\eta^I(m,t)$ for $\eta \in \{\emptyset, G, B\}$ and $m \in \{G, B\}$;

2. $\tilde{r}_\eta^C(m,t) = 1$ for $m = \eta$, $\tilde{r}_\eta^C(m,t) = 0$ for $m \neq \eta$, and $\tilde{r}_\eta^I(m,t) = \frac{r^I(m,t)}{\tilde{r}^I_\eta(m,t)}$ for all $t \in S\setminus \{\bar{t}\}$.

By Proposition 9 a strategy profile constitutes a delay equilibrium on $S$ with $|S| \geq 3$ only if $b_t = b(G,t) = b(B,t) = b_0$ for all $t \in S\setminus \{\bar{t}\}$. Therefore $r^I(m,t) = r^C(m,t)$ for $m \in \{G, B\}$ and all $t \in S\setminus \{\bar{t}\}$. Consequently, $\tilde{r}_\eta^I(m,t) = \frac{r^I(m,t)}{\tilde{r}^I_\eta(m,t)} < \frac{r^I(m,t)}{\tilde{r}^I_\eta(m,t)} = 1$, that is $\tilde{r}_\eta^C(m,t) = 1$ is indeed a well-defined profile of strategies. Moreover, profile $\{\tilde{r}^G_\eta(m,t)\}$ induces the same beliefs as profile $\{r_\eta^G(m,t)\}$ after the same histories, and therefore also constitutes an equilibrium. At the same time, this equilibrium is a relay one because $r^G_\eta(G,m) = r^G_\eta(B,m) = 1$.

Next we proceed with describing which conditions are necessary for a given profile of strategies $\{r_\eta^G(m,t)\}$ to constitute a relay equilibrium. We consider two sub-cases depending on whether not making a report by $\bar{t}$ is on equilibrium path.

Lemma 10. Suppose that beliefs $b(m,t)$ and $b^r(m,t)$ for all $t \in S$ are given by (13) and (14) respectively, while $b_t$ is given by (15) for all $t < \bar{t}$. Moreover, let strategy profile $\{r_\eta^G(m,t)\}$ be such that: (1) $r_\eta^C(\bar{t},m) = 1$ for all $\eta \in \{G, B\}$ and all $t \in S$, and (2) $r_\eta^I(m,t) = 0$ for all $m \in \{G, B\}$ and all $t \in S\setminus \{\bar{t}\}$.

1. Strategy profile $\{r_\eta^G(m,t)\}$ with $r^G_\eta(G,\bar{t}) + r^G_\eta(B,\bar{t}) = 1$ constitutes a relay equilibrium on $S$ only if

$$
W^I_t(m,t) = \tilde{W} \text{ for all } t \in S \text{ and } m \in \{G, B\} \text{ for some } \tilde{W} \in \mathbb{R}_+ \text{,}
$$

and $r^I(G,\bar{t}) + r^I(B,\bar{t}) = 1$.

Moreover, there exists at most one solution to this system, and if $w(x) = x^a$ then this solution always exists.
2. If \(16\) holds then strategy profile \(\{r^*_m(m,t)\}\) with \(r^*_G(G,t) + r^*_B(B,t) < 1\) constitutes a relay equilibrium on \(S\) only if

\[
W^*_t(m,t) = W^*_t(\emptyset) = W \text{ for all } t \in S \text{ and } m \in \{G,B\} \text{ for some } W \in \mathbb{R}_+.
\]

Moreover, there exists at most one solution to this system, and if \(w(x) = x^n\) then this solution always exists.

Proof. By Proposition \(1\) a strategy profile constitutes an equilibrium only if \(W^*_t(m,t)\) is constant for all \(t \in S \text{ and } m \in \{G,B\}\). Additionally, if \(r^*_G(G,t) + r^*_B(B,t) < 1\) – that is, not making a report by \(t\) is an on-path action – the value that the incompetent expert receives from making any report must be equal to value from making no report.

The proof of Proposition \(1\) argued that \(r^C(G,t) + r^C(B,t) = 1\) implies \(r^I(G,t) + r^I(B,t) = 1\). From Proposition \(2\) we know that \(\{r^*_m(m,t)\}\), values of making a report \(W^*_t(m,\tau)\), \(\bar{W}^*_t(m,\tau)\), and belief profiles \(\tilde{b}, \tilde{b}\) that solve either system \(16\) or system \(19\). Then \(r^I(G,t_1) \neq \tilde{r}^I(G,t_1)\), as otherwise equilibria coincide. Indeed, strategies \(r^*_m(m,t) = \tilde{r}^*_m(m,t)\) for all \(t \in S\). Therefore, if \(r^I(G,t_1) = \tilde{r}^I(G,t_1)\) then \(b(G,t_1) = \tilde{b}(G,t_1)\) and \(b^+(G,t_1) = \tilde{b}^+(G,t_1)\), meaning that \(W^*_t(G,t_1) = \tilde{W}^*_t(G,t_1)\). By the first two parts of the lemma, incompetent expert’s values \(W^*_t(m,t)\) should then coincide for all \(m \in S\), which implies \(r^I(m,t) = \tilde{r}^I(m,t)\) – a contradiction.

Without loss, assume \(r^I(G,t_1) > \tilde{r}^I(G,t_1)\). Then since \(W^*_t(G,t_1) = \tilde{W}^*_t(G,t_1)\) and \(\bar{W}^*_t(G,t_1) = \tilde{W}^*_t(G,t_1)\), we must have \(r^I(B,t_1) > \tilde{r}^I(B,t_1)\) as well. By \(15\) this implies that \(b_1 > \tilde{b}_1\). Consequently, \(r^I(G,t_2) > \tilde{r}^I(G,t_2)\) and \(r^I(B,t_2) > \tilde{r}^I(B,t_2)\) because \(W^*_t(m,t_1) = W^*_t(m,t_2)\) and \(\tilde{W}^*_t(m,t_1) = \tilde{W}^*_t(m,t_2)\) for \(m \in \{G,B\}\). Iterating this logic further, we obtain that \(r^I(G,t) + r^I(B,t) > \tilde{r}^I(G,t) + \tilde{r}^I(B,t)\). In the context of the first part of the lemma (case \(r^*_G(G,t) + r^*_B(B,t) = 1\)), it clearly violates \(r^I(G,t) + r^I(B,t) = \tilde{r}^I(G,t) + \tilde{r}^I(B,t) = 1\). In the context of the second part, it implies \(b_t > \tilde{b}_t\), and therefore \(W^*_t(\emptyset) > \tilde{W}^*_t(\emptyset)\) because the payoff that the incompetent expert receives from staying silent is point-wise lower in the former equilibrium. At the same time, because \(r^I(G,t_1) > \tilde{r}^I(G,t_1)\), we must have \(W^*_t(G,t_1) < \tilde{W}^*_t(G,t_1)\). As in the second case \(W^*_t(\emptyset) = W^*_t(G,t_1)\) and \(\tilde{W}^*_t(\emptyset) = \tilde{W}^*_t(G,t_1)\), we arrive to a contradiction.

Finally, to prove existence of a solution for \(w(x) = x^n\) assume first that \(r^*_G(G,t) + r^*_B(B,t) = 1\). The first part of the lemma then implies that we have \(W^*_t(G,t) = W^*_t(B,t)\) and \(r^I(G,t) = r^I(B,t) = 1\). Taking into account \(w(x) = x^n\), \(13\) and \(14\), we can explicitly solve this system of equations for \(r^I(G,t)\) and \(r^I(B,t)\) as functions of \(r^*_G(G,t)\) and \(r^*_B(B,t)\). The resulting expressions are

\[
\begin{align*}
r^I(G,t) &= \frac{M_G(r^*_G(G,t))^\frac{1}{n}}{M_G(r^*_G(G,t))^{\frac{1}{n}} + M_B(r^*_B(B,t))^{\frac{1}{n}}}, \\
r^I(B,t) &= \frac{M_B(r^*_B(B,t))^\frac{1}{n}}{M_G(r^*_G(G,t))^{\frac{1}{n}} + M_B(r^*_B(B,t))^{\frac{1}{n}}},
\end{align*}
\]

where

\[
M_G(x) := (T - \tilde{t}) \cdot (p_0 (\tilde{z} + (1 - \tilde{z}) x) + \theta p_0 (\tilde{z} + (1 - \tilde{z}) x) + \theta (1 - p_0) ((1 - \rho) \tilde{z} + (1 - \tilde{z}) x)) ,
\]

\[
M_B(x) := (T - \tilde{t}) \cdot ((1 - p_0) (\tilde{z} + (1 - \tilde{z}) x) + \theta p_0 ((1 - \rho) \tilde{z} + (1 - \tilde{z}) x) + \theta (1 - p_0) (\rho \tilde{z} + (1 - \tilde{z}) x)) .
\]

In case \(r^*_G(G,t) + r^*_B(B,t) < 1\), the second part of the lemma prescribes that \(W^*_t(G,t) = W^*_t(B,t) = \frac{M_G(r^*_G(G,t))^{\frac{1}{n}}}{M_G(r^*_G(G,t))^{\frac{1}{n}} + M_B(r^*_B(B,t))^{\frac{1}{n}}}\).
Consider strategy profile given by (21). Analogously to the previous case, we can solve for $r^I(G, \bar{t})$ and $r^I(B, \bar{t})$ as functions of $r^C_G(G, \bar{t})$ and $r^C_B(B, \bar{t})$ and obtain
\begin{align*}
r^I(G, \bar{t}) &= \frac{M_G (r^C_G(G, \bar{t}))^{\frac{1}{n}}}{M_G (r^C_G(G, \bar{t}))^{\frac{1}{n}} + M_B (r^C_B(B, \bar{t}))^{\frac{1}{n}} + (1 - \bar{z}) \cdot (T - \bar{t} + \theta) \cdot (1 - r^C_G(G, \bar{t}) - r^C_B(B, \bar{t}))}, \\
r^I(B, \bar{t}) &= \frac{M_B (r^C_B(B, \bar{t}))^{\frac{1}{n}}}{M_G (r^C_G(G, \bar{t}))^{\frac{1}{n}} + M_B (r^C_B(B, \bar{t}))^{\frac{1}{n}} + (1 - \bar{z}) \cdot (T - \bar{t} + \theta) \cdot (1 - r^C_G(G, \bar{t}) - r^C_B(B, \bar{t}))}.
\end{align*}

(21)

Note that expressions in (20) can be obtained from the respective ones in (21) substituting $r^C_G(G, \bar{t}) + r^C_B(B, \bar{t}) = 1$. Therefore, without loss we can restrict ourselves to the case $r^C_G(G, \bar{t}) + r^C_B(B, \bar{t}) < 1$ and only consider strategy profile given by (21). All above proves existence of the solution for $t = \bar{t}$.

We establish existence for all $t \in S \setminus \{\bar{t}\}$ proceeding by backward induction. Consider system
\begin{equation*}
W^I_t (\emptyset) (G, t | S | t-1) = W^I_{t, t-1} (B, t | S | t-1) = W^I_{\bar{t}, t-1} (G, \bar{t})
\end{equation*}

We next show that this system of equations always has a solution. Consider the following auxiliary system. For any given $c > 0$ assume that $r^I(G, t | S | t-1) + r^I(B, t | S | t-1) = c$ and consider equation
\begin{equation*}
W^I_{t, t-1} (G, t | S | t-1) = W^I_{t, t-1} (B, t | S | t-1).
\end{equation*}

Then if $r^I(G, t | S | t-1)$ approaches zero, the LHS approaches $+\infty$ while the RHS is constant. Similarly, the RHS strictly dominates the LHS when $r^I(G, t | S | t-1) = c$. Moreover the LHS is strictly decreasing in $r^I(G, t | S | t-1)$, while the RHS is strictly increasing in it. Therefore by the Intermediate Value Theorem for a given $c > 0$ there exists a unique pair $r^I(G, t | S | t-1), r^I(B, t | S | t-1)$ such that $r^I(G, t | S | t-1) + r^I(B, t | S | t-1) = c$ and $W^I_{t, t-1} (G, t | S | t-1) = W^I_{t, t-1} (B, t | S | t-1)$. Also note that both $r^I(G, t | S | t-1)$ and $r^I(B, t | S | t-1)$ are strictly increasing in $c$. Further for the same $c > 0$ still assume that $r^I(G, t | S | t-1) + r^I(B, t | S | t-1) = c$ and consider equality
\begin{equation*}
W^I_{t, t-1} (G, t | S | t-1) = W^I_{t, t-1} (G, \bar{t})
\end{equation*}
as an equation in $c$. The RHS of it is a strictly increasing function of $c$ which approaches $+\infty$ when $c$ approaches 1. As established before, the LHS of it is a strictly decreasing function of $c$ (because $r^I(G, t | S | t-1)$ is strictly increasing in $c$), which approaches $+\infty$ when $c$ approaches zero. Therefore there exist unique $r^I(G, t | S | t-1)$ and $r^I(B, t | S | t-1)$ such that $W^I_{t, t-1} (G, t | S | t-1) = W^I_{t, t-1} (B, t | S | t-1) = W^I_{t, t-1} (G, \bar{t})$, which finishes the proof.

The bottom line of the lemma above is that for a given tuple $[S, r^C_G(G, \bar{t}), r^C_B(B, \bar{t})]$, a strategy profile that constitutes a relay equilibrium is a unique solution to a particular system of algebraic equations. Moreover, solution to this system always exists if $w(x) = x^n$. Representing a strategy profile as a solution to a system of equations allows us to compare equilibrium strategies and, therefore, report informativeness across different relay equilibria employing the arguments similar to the Implicit Function Theorem.

In all further lemmas it is assumed that strategy profile $r^G (m, \tau)$ and all associated equilibrium objects such as values $W^I_t (m, \tau)$, belief profiles $b, p$, and informativeness measures $i(m, \tau)$ constitute a solution to either system (18) or system (19) for a given tuple $[S, r^C_G(G, \bar{t}), r^C_B(B, \bar{t})]$, and therefore are understood as functions of $[S, r^C_G(G, \bar{t}), r^C_B(B, \bar{t})]$.

The next lemma establishes that whenever $w(x) = x^n$, it is true that strategies that constitute a solution to either system (18) or system (19) are continuously differentiable in $r^C_G(G, \bar{t})$ and $r^C_B(B, \bar{t})$ at
\( r^C_G(G, \bar{t}) + r^C_B(B, \bar{t}) = 1 \). The same will then be true of all associated equilibrium objects \( W^I_I(m, \tau), b \) and \( i(m, t) \), as they all are continuously differentiable functions of the strategies. The statement of this lemma is valid for any continuously differentiable \( w(x) \), but the statement for the case \( w(x) = x^a \) is enough for the needs of the paper and is significantly easier to prove. Lemma 11 allows us to further omit the consideration of the case \( r^C_G(G, \bar{t}) + r^C_B(B, \bar{t}) = 1 \) and without loss assume in further propositions that \( r^C_G(G, \bar{t}) + r^C_B(B, \bar{t}) < 1 \).

**Lemma 11.** Suppose \( w(x) = x^a \), and strategy profile \( \{r^*_m(m, t)\} \) solves either system (18) or system (19). Then \( r^*_m(m, t) \) is a continuously differentiable function of \( r^C_G(G, \bar{t}) \) and \( r^C_B(B, \bar{t}) \) for all \( r^C_G(G, \bar{t}) + r^C_B(B, \bar{t}) < 1 \).

**Proof.** First note that \( r^*_m(m, t) \) exists by Lemma 19. Next, the strategy profile for competent expert is fixed by the premise of Lemma 10 and is therefore a continuously differentiable function of \( r^C_G(G, \bar{t}) \) and \( r^C_B(B, \bar{t}) \). Therefore we are left to establish that \( r^I(m, t) \) is a continuously differentiable function of \( r^C_G(G, \bar{t}) \) and \( r^C_B(B, \bar{t}) \) for all \( m \in \{G, B\} \) and all \( t \in S \). As expressions in (20) coincide with the ones in (21) for \( r^C_G(G, \bar{t}) + r^C_B(B, \bar{t}) = 1 \), without loss we restrict ourselves to the case \( r^C_G(G, \bar{t}) + r^C_B(B, \bar{t}) < 1 \). Both expressions in (21) are continuously differentiable functions of \( r^C_G(G, \bar{t}) \) and \( r^C_B(B, \bar{t}) \) for \( r^C_G(G, \bar{t}) + r^C_B(B, \bar{t}) \leq 1 \). Thus, it is left to show the same for \( r^I(G, t) \) and \( r^I(B, t) \) for \( t \in S \setminus \{\bar{t}\} \). We proceed using backward induction. Consider two equalities

\[
\begin{align*}
W^I_{t[S\setminus t]}(G, t[S\setminus t]) &= W^I_I(G, \bar{t}), \\
W^I_{t[S\setminus t]}(B, t[S\setminus t]) &= W^I_I(B, \bar{t}).
\end{align*}
\]

Given \( r^I(G, \bar{t}) \) and \( r^I(B, \bar{t}) \), they constitute a system of equations on \( r^I(G, t[S\setminus t]) \) and \( r^I(B, t[S\setminus t]) \). Moreover, because \( w(x) = x^a \), both \( r^I(G, t[S\setminus t]) \) and \( r^I(B, t[S\setminus t]) \) do not depend on \( b_{t[S\setminus t]} \) – that is, on strategies \( r^I(G, t) \) and \( r^I(B, t) \) for \( t \leq t[S\setminus t] \) – but only on \( r^I(G, \bar{t}) \) and \( r^I(B, \bar{t}) \). Therefore, by the Implicit Function Theorem, \( r^I(G, t[S\setminus t]) \) and \( r^I(B, t[S\setminus t]) \) are continuously differentiable functions of \( r^I(G, \bar{t}) \) and \( r^I(B, \bar{t}) \), which eventually implies that they are continuously differentiable functions of \( r^C_G(G, \bar{t}) \) and \( r^C_B(B, \bar{t}) \). Proceeding backwards we establish the claim for all \( r^I(G, t) \) and \( r^I(B, t) \) for \( t \in S \).

The two following lemmas are mostly technical and carry little intuition.

**Lemma 12.** Suppose

\[
m(x) = (\beta_1(a_1 + bx)^\alpha + \cdots + \beta_k(a_k + bx)^\alpha)\frac{1}{b} - bx
\]

for \( k \geq 2, b > 0, \sum_{i=1}^{k} \beta_i = 1, a_1, \ldots, a_k \geq 0 \) with \( a_i, a_j > 0 \) for some \( i, j \in \{1, \ldots, k\}, i \neq j \). Then \( m(x) \) is strictly decreasing when \( \alpha > 1 \) and is strictly increasing when \( \alpha < 1 \).

**Proof.** Begin by observing that

\[
\frac{1}{b} \frac{dm(x)}{dx} = \frac{\beta_1(a_1 + bx)^{\alpha-1} + \cdots + \beta_k(a_k + bx)^{\alpha-1}}{(\beta_1(a_1 + bx)^\alpha + \cdots + \beta_k(a_k + bx)^\alpha)^{\frac{\alpha}{\alpha-1}}} - 1.
\]

First, if \( \alpha > 1 \) then, since \( x^k \) is strictly convex for \( k > 1 \), we have

\[
(\beta_1(a_1 + bx)^{\alpha-1} + \cdots + \beta_k(a_k + bx)^{\alpha-1})^{\frac{1}{\alpha-1}} < (\beta_1(a_1 + bx)^\alpha + \cdots + \beta_k(a_k + bx)^\alpha)^\frac{1}{\alpha},
\]

and therefore \( \frac{dm(x)}{dx} < 0 \) if \( \alpha > 1 \).
Second, if $\alpha < 1$ then, because $x^k$ is strictly convex for $k < 0$, we still have \( \frac{\alpha - 1}{\alpha} < 0 \), we have

$$\beta_1 (a_1 + bx)^{\alpha - 1} + \ldots + \beta_k (a_k + bx)^{\alpha - 1} > (\beta_1 (a_1 + bx)^\alpha + \ldots + \beta_k (a_k + bx)^\alpha) \frac{\alpha - 1}{\alpha} ,$$

and therefore $\frac{d n(x)}{dx} > 0$ if $\alpha < 1$.

\[\square\]

**Lemma 13.** Suppose $w(x) = x^\alpha$, and strategy profile \( \{\tilde{r}_i^j(m,t)\} \) solves either system \( (18) \) or system \( (19) \). Then if $\alpha < 1$ for any $m \in \{G, B\}$ we have

$$\frac{\partial}{\partial r^j(G,t)} \left( \frac{(1 - \tilde{t}_i) \cdot (1 - r^j(G,t) - r^j(B,t))}{1 - r^j(G,t) - r^j(B,t)} \right) > 0,$$

while if $\alpha > 1$ for any $m \in \{G, B\}$ we have

$$\frac{\partial}{\partial r^j(G,t)} \left( \frac{(1 - \tilde{t}_i) \cdot (1 - r^j(G,t) - r^j(B,t))}{1 - r^j(G,t) - r^j(B,t)} \right) < 0.$$

Additionally, for any $m \in \{G, B\}$ and any $t \in S$ we have that $W_t^j(m,t)$ is strictly increasing in $r^j(G,t)$ if $\alpha < 1$, and $W_t^j(m,t)$ is strictly decreasing in $r^j(B,t)$ if $\alpha > 1$.

**Proof.** Lemmas \[10\] and \[11\] imply that $r^j(m,t)$ exists for all $m \in \{G, B\}$ and $t \in S$ and is continuously differentiable in $r^j(G,t)$ and $r^j(B,t)$. Next, from \[21\] we can calculate that

$$\frac{(1 - \tilde{t}_i) \cdot (1 - r^j(G,t) - r^j(B,t))}{1 - r^j(G,t) - r^j(B,t)} = M_G (r^j(G,t))^\frac{\alpha}{2} + M_B (r^j(B,t))^\frac{\alpha}{2} + (1 - \tilde{t}_i) \cdot (1 - r^j(G,t) - r^j(B,t)) ,$$

which is the sum of two functions of the form from Lemma \[12\] and a constant. Therefore, the first statement of the Lemma follows directly from Lemma \[12\].

Next we establish the second claim. By Lemma \[11\] we can assume without loss that $r^j(G,t) + r^j(B,t) < 1$. What follows is the proof for the case $\alpha < 1$ (case $\alpha > 1$ is analogous). For a given $m \in \{G, B\}$ fix some $r^j(G,t) < r^j(m,t)$ and $r^j(B,t) = r^j(m,t)$. Also denote the respective strategy profiles that solve system \( (19) \) for $\{S, r^j(G,t), r^j(B,t)\}$ and $\{S, r^j(G,t), r^j(B,t)\}$ as $\{\tilde{r}^j(m,t)\}$ and $\{\tilde{r}^j(m,t)\}$. Denote by $b$ and $\tilde{b}$ the respective beliefs, and by $W_t^j(m,\tau)$ and $\tilde{W}_t^j(m,\tau)$ the respective values from reports.

Assume that $W_t^j(m,t) > \tilde{W}_t^j(m,t)$. Then $r^j(m,t_1) \leq \tilde{r}^j(m,t_1)$ for $m \in \{G, B\}$. From \[15\] we then get that $b_{t_1} \leq \tilde{b}_{t_1}$. This, in turn, implies that $r^j(G,t_2) \leq \tilde{r}^j(G,t_2)$ and $r^j(B,t_2) \leq \tilde{r}^j(B,t_2)$ because $W_t^j(m,t_1) = W_t^j(m,t_2)$ and $\tilde{W}_t^j(m,t_1) = \tilde{W}_t^j(m,t_2)$ for $m \in \{G, B\}$. Iterating this logic further, we get $b_{t_{1,\ldots,1}} \leq \tilde{b}_{t_{1,\ldots,1}}$.

From the first part of the lemma we know that

$$\frac{(1 - \tilde{t}_i) \cdot (1 - r^j(G,t) - r^j(G,t))}{1 - r^j(G,t) - r^j(B,t)} > \frac{(1 - \tilde{t}_i) \cdot (1 - r^j(G,t) - r^j(G,t))}{1 - r^j(G,t) - r^j(B,t)} ,$$

and therefore $W_t^j(\emptyset) < \tilde{W}_t^j(\emptyset)$. This gives us a contradiction with the initial assumption $W_t^j(m,t) > \tilde{W}_t^j(m,t)$ because we must have $W_t^j(\emptyset) = W_t^j(G,t_1)$ and $\tilde{W}_t^j(\emptyset) = \tilde{W}_t^j(G,t_1)$.

\[\square\]

**Proof of Proposition 4** Part 1. We first show that for any set of parameters $|S| = 1$, $r^j(G,t) > 0$, and $r^j(B,t) > 0$, the informative equilibrium with given parameters exists. Proposition \[2\] and the values of $r^j(m,t)$ pin down the competent expert’s strategy. We next show that there exists such an incompotent
expert’s strategy $r^I(m, l)$ that conditions in Lemma 10 are satisfied, which proves this part of the proposition. Note also that for singleton $S$ we have $\bar{z}_l = F(l)$.

The first condition one needs to check in order to establish existence is $W^I_l(G, l) = W^I_l(B, l)$, which can be written as

$$(T - \bar{t}) \cdot w \left( \frac{b(G, l)}{1 - b(G, l)} \right) + \theta \cdot \left( p_0 w \left( \frac{b^C(G, l)}{1 - b^C(G, l)} \right) + (1 - p_0) w \left( \frac{b^P(G, l)}{1 - b^P(G, l)} \right) \right) = (T - \bar{t}) \cdot w \left( \frac{b(B, l)}{1 - b(B, l)} \right) + \theta \cdot \left( p_0 w \left( \frac{b^C(B, l)}{1 - b^C(B, l)} \right) + (1 - p_0) w \left( \frac{b^P(B, l)}{1 - b^P(B, l)} \right) \right).$$

(23)

From (13) and (14) we see that the LHS is strictly decreasing in $r$ in that (23) is satisfied and

$$(T - \bar{t}) \cdot w \left( \frac{b(G, l)}{1 - b(G, l)} \right) + \theta \cdot \left( p_0 w \left( \frac{b^C(G, l)}{1 - b^C(G, l)} \right) + (1 - p_0) w \left( \frac{b^P(G, l)}{1 - b^P(G, l)} \right) \right) = (T - \bar{t}) \cdot w \left( \frac{b(B, l)}{1 - b(B, l)} \right) + \theta \cdot \left( p_0 w \left( \frac{b^C(B, l)}{1 - b^C(B, l)} \right) + (1 - p_0) w \left( \frac{b^P(B, l)}{1 - b^P(B, l)} \right) \right).$$

(24)

By the same logic as above, we know that for any given $c > 0$ there exist unique $r^I(G, l)$ and $r^I(B, l)$ such that (23) is satisfied and $r^I(G, l) + r^I(B, l) = c$. Further, note that $r^G_{\bar{G}}(G, l)$ and $r^G_{\bar{G}}(B, l)$ that are obtained as a solution to this auxiliary system are both increasing in $c$.

Finally, consider (24) as an equation in $c$. From the previous observation it follows that its LHS is decreasing in $c$, while the RHS is increasing in $c$. If $c = 0$ then the LHS dominates the RHS, and if $c = 1$ the RHS dominates the LHS. Therefore, by the Intermediate Value Theorem there exists unique $c$ such that (24) is satisfied. Solving (23) using this $c$ gives a pair $r^I(G, l)$ and $r^I(B, l)$ that uniquely solves the original system of (23) and (24).

**Part 2.** To prove the second part of the proposition we construct a relay equilibrium for a given $S$ with $|S| \geq 2$ and $r^G_{\bar{G}}(G, l) = r^G_{\bar{G}}(B, l) = 0$. Since states are symmetric, for any $t \in S$ the incompetent expert is indifferent between reports $(G, t)$ and $(B, t)$ if and only if $r^I(G, t) = r^I(B, t)$. Thus, we are only left to ensure the indifference between a report and no report for incompetent expert and to verify that the constructed equilibrium is incentive compatible for informed competent expert. Define $g := g(G, t_1) = g(B, t_1)$. Then

$W^I_l(G, t_1) = W^I_l(B, t_1)$ is equal to

$$(T - \bar{t}) \cdot w \left( \frac{b_0}{1 - b_0} \cdot e^g \right) + \theta \cdot \left( p_0 w \left( \frac{b_0}{1 - b_0} \cdot \frac{p_0 \theta}{p_0} \right) + (1 - p_0) w \left( \frac{b_0}{1 - b_0} \cdot \frac{(1 - \rho) e^g}{p_0} \right) \right).$$

From the expression above we see that the value of $g$ fully determines the value that the incompetent expert gets in equilibrium. In a relay equilibrium the competent expert’s strategy is fixed, so smaller $g$ means larger $r^I(G, t_1)$ and $r^I(B, t_1)$. Larger $r^I(G, t_1)$ and $r^I(B, t_1)$, in turn, imply higher $b_1$. Finally, because incompetent expert must be indifferent between reports at $t_1$ and $t_2$, higher $b_1$ implies larger $r^I(G, t_2)$ and $r^I(B, t_2)$. All in all, it means that the payoff that incompetent expert receives by not making a report is point-wise strictly decreasing in $g$. When $g = 0$ we have that $b_1 = b_0$ for all $t \in T$ (remember that $r^C_{\bar{G}}(G, l) = r^C_{\bar{G}}(B, l) = 0$, so following the logic from Proposition 8 we have $g(m, l) = 0$ for all $t \in T$), therefore value of making no

\[24\] It is clear that at least one of $r^G_{\bar{G}}(G, l)$ and $r^G_{\bar{G}}(B, l)$ must be higher for a higher $c$, and (23) implies higher $r^G_{\bar{G}}(B, l)$. 38
report by the end of period $T$ evaluated at $t_1$ is equal to $(T - t_1 + \theta) \cdot w \left( \frac{b_0}{1 - b_0} \right)$. When $g \to -\infty$ we have that the value of making no report strictly dominates the value of making a report. When $g = 0$ we have

$$(T - t_1) \cdot w \left( \frac{b_0}{1 - b_0} \right) + \theta \cdot \left[ p_0 w \left( \frac{b_0}{1 - b_0} \cdot \frac{\rho}{\bar{p}_0} \right) + (1 - p_0) w \left( \frac{b_0}{1 - b_0} \cdot \frac{1 - \rho}{\bar{p}_0} \right) \right] \geq (T - t_1 + \theta) \cdot w \left( \frac{b_0}{1 - b_0} \right)$$

because $w(x)$ is convex. Therefore, by the Intermediate Value Theorem, there exists a unique $g \leq 0$ such that the incompetent expert is indifferent between making a report and not making a report. Finally, because $g(G, t_1) = g(B , t_1) = g = \leq 0$ we have $b(m , t_1) = b_0 \leq b_t$. From Proposition 5 for convex $w(x)$ we know that it implies that $b(m , t) - \eta$ and, consequently, $\Delta w_\eta(m , t)$ for $m = \eta$ are decreasing on $S$, which verifies that $r_G^m(\eta , t_1) = 1$ is an optimal strategy for the competent expert, i.e., the constructed profile indeed constitutes an equilibrium.

**Part 3.** To prove the third part assume the contrary: there exists $S$ with $\mid S \mid \geq 3$ such that informative equilibrium with respective strategy profile $\{r_G^m(\eta , t_1)\}$ for $t \in S$ exists. By Lemma 10 we can assume without loss that the equilibrium is a relay one. By Lemma 10 we know that there exists a strategy profile $\{r_G^m(\eta , t_1)\}$ (and associated belief profile $\tilde{b}$ and value function $W^\tilde{b}(m , \tau)$) for the same $S$ with $r_G^m(\tau , t_1) = r_G^m(B , t_1) = 0$ which solves system (19). We next show that this profile constitutes a relay equilibrium on $S$. The only condition that needs to be verified is that this profile is incentive compatible for informed competent expert. By the proof of Proposition 5 for $S$ with $\mid S \mid \geq 3$ this is equivalent to verifying that $\tilde{b}_1 \geq b(m , t_1)$ for $m \in \{ G , B \}$ because $r_G^m(\tau , t_1) = r_G^m(B , t_1) = 0$.

In the original equilibrium we have $b_1 \geq b(m , t_1)$ by the same Proposition 5. By Lemma 10 because $r_G^m(\tau , t) \geq r_G^m(m , t)$ for $m \in \{ G , B \}$ and $\alpha < 1$, we have $W^\tilde{b}_1(m , t_1) > W^\tilde{b}_1(m , t_1)$. This implies that $r(m , t_1) < \tilde{r}(m , t_1)$ for $m \in \{ G , B \}$, and therefore $\tilde{b}_1 > b_1 \geq b(m , t_1) > \tilde{b}(m , t_1)$ for $m \in \{ G , B \}$, which completes the argument.

We have established the existence of the relay equilibrium on $S$ with $r_G^m(\tau , t_1) = r_G^m(B , t_1) = 0$. By Proposition 5 there exists $m \in \{ G , B \}$ such that $\tilde{b}(m , t_1) \leq b_0$, and therefore $W^\tilde{b}_1(m , t_1) \leq (T - t_1) \cdot w \left( \frac{b_0}{1 - b_0} \right) + \theta \cdot \left[ p_0 w \left( \frac{b_0}{1 - b_0} \cdot \frac{\rho}{\bar{p}_0} \right) + (1 - p_0) w \left( \frac{b_0}{1 - b_0} \cdot \frac{1 - \rho}{\bar{p}_0} \right) \right]$. At the same time, because in such equilibrium $b_t \geq b_0$ for all $t \in S$ (again by Proposition 5), we have $W^\tilde{b}_1(\emptyset) \geq (T - t_1 + \theta) \cdot w \left( \frac{b_0}{1 - b_0} \right)$. Finally, $W^\tilde{b}_1(m , t_1) = W^\tilde{b}_1(\emptyset)$ implies

$$p_0 w \left( \frac{b_0}{1 - b_0} \cdot \frac{\rho}{\bar{p}_0} \right) + (1 - p_0) w \left( \frac{b_0}{1 - b_0} \cdot \frac{1 - \rho}{\bar{p}_0} \right) \geq w \left( \frac{b_0}{1 - b_0} \right). \quad (25)$$

If $w(x)$ is strictly concave then (25) can not be satisfied, which gives us the contradiction. \qed

**Proof of Proposition 5** Denote by $W^b(m , \tau)$ and $W^\tilde{b}(m , \tau)$ the respective values of making report and by $\tilde{b}$ and $\tilde{b}$ the beliefs for strategy profiles $\{r_G^m(\tau , t_1)\}$ and $\{r_G^m(t_1)\}$.

To prove the first part of the proposition, we first show that $W^b_1(m , t_1) \leq W^\tilde{b}_1(m , t_1)$ for $m \in \{ G , B \}$. Assume the contrary, i.e., that $W^b_1(m , t_1) > W^\tilde{b}_1(m , t_1)$ for $m \in \{ G , B \}$. Then it directly implies $r^b_1(m , t_1) < \tilde{r}^\tilde{b}_1(m , t_1)$ for $m \in \{ G , B \}$. From (15) it then follows that $b_1 < \tilde{b}_1$. This, in turn, implies that $W^b_1(G , t_1) = W^b_1(B , t_1)$ and $W^\tilde{b}_1(G , t_1) = W^\tilde{b}_1(B , t_1)$. We have that either $W^b_1(m , t_1) > W^\tilde{b}_1(m , t_1)$ for both $m \in \{ G , B \}$ or $W^b_1(m , t_1) < W^\tilde{b}_1(m , t_1)$ for both $m \in \{ G , B \}$.

25Because $W^b_1(G , t_1) = W^b_1(B , t_1)$ and $W^\tilde{b}_1(G , t_1) = W^\tilde{b}_1(B , t_1)$, we have that either $W^b_1(m , t_1) > W^\tilde{b}_1(m , t_1)$ for both $m \in \{ G , B \}$ or $W^b_1(m , t_1) < W^\tilde{b}_1(m , t_1)$ for both $m \in \{ G , B \}$. 39
the flow payoff from staying silent until $t$ is a martingale. Therefore, $W_{t_1}^I(m,t) < W_{t_1}^I(m,t_1)$, which gives us a contradiction with the initial assumption.

Condition $W_{t_1}^I(m,t) \leq \tilde{W}_{t_1}^I(m,t)$ directly implies that $r^I(m,t_1) \geq r^I(m,t_1)$ – since in a relay equilibrium the strategy of competent expert is fixed, and therefore $|i(m,t)| \leq |\tilde{i}(m,t)|$ for all $t \in S$. Finally, $|i(m,t)| = 0$ for any $t \notin S$, meaning that $|i(m,t)| < |\tilde{i}(m,t)|$ for $t \in S \setminus S$.

The second part of the proposition for $S$ with $|S| \geq 3$ follows directly from Lemma 13. Indeed, Lemma 9 implies that we can consider only relay equilibria, and in this case Lemma 13 implies that $W_{t_1}^I(m,t) < \tilde{W}_{t_1}^I(m,t)$ if $\alpha < 1$, and that $W_{t_1}^I(m,t) > \tilde{W}_{t_1}^I(m,t)$ if $\alpha > 1$ for all $t \in S \setminus \{1\}$. Analogously to the first part, these conditions directly imply that $|i(m,t)| < |\tilde{i}(m,t)|$ for all $t \in S \setminus \{1\}$ if $\alpha < 1$, and that $|i(m,t)| > |\tilde{i}(m,t)|$ for all $t \in S \setminus \{1\}$ if $\alpha > 1$.

For $S$ with $|S| = 2$ we still have $r^C_\eta(m,t) = 1$, and therefore [20] and [21] remain valid. Therefore Lemma 13 remains valid as well, which implies that $W_{t_1}^I(m,t_1) < W_{t_1}^I(m,t_1)$ if $\alpha < 1$, and that $W_{t_1}^I(m,t_1) > \tilde{W}_{t_1}^I(m,t_1)$ if $\alpha > 1$. Finally, $W_{t_1}^I(m,t_1) < W_{t_1}^I(m_1)$ implies that either $b(m,t_1) < \tilde{b}(m_1)$, or $b^m(m_1) < \tilde{b}^m(m_1)$ or $\tilde{b}^{-m}(m_1) < \tilde{b}^{-m}(m_1)$. However, $b(m,t_1)$, $b^m(m_1)$ and $\tilde{b}^{-m}(m_1)$ are scalar multipliers of each other as well as $\tilde{b}(m_1)$, $\tilde{b}^m(m_1)$ and $\tilde{b}^{-m}(m_1)$. Therefore all three inequalities hold simultaneously. Therefore $b^m(m_1) < \tilde{b}^m(m_1)$. Finally, remember that $|i(m,t_1)|$ is a strictly increasing function of $b^m(m_1)$. This directly implies that $|i(m,t_1)| < |\tilde{i}(m,t_1)|$ if $\alpha < 1$, and that $|i(m,t_1)| > |\tilde{i}(m,t_1)|$ if $\alpha > 1$.

Proofs for Section 5

Proof of Proposition 7: In an ideal equilibrium, $r^C_\eta(m,t) = 1$ and $r^C_\eta(m,t) = r^I(m,t) = 0$ for $\eta, m \in \{G, B\}$ and all $t \in S$. Report $(\eta, t)$ at $t$ yields maximal continuation reputation to the competent expert with information $\eta \in \{G, B\}$, so is trivially optimal at the time when he receives his private signal. Preferences of the uninformed competent expert coincide (ordinally) at any $t$ with those of an incompetent expert. Therefore, it is enough to verify that the incompetent expert prefers to stay silent at every point of the support. Since after any report the reputation jumps to $\lim_{x \to +\infty} w(x)$, in case of no report it must decrease, as $b_t$ is a martingale. Therefore, $W_{t_1}^I(m,t)$ is maximized at $t = t_1$ (and any $m$). Report $(G,t_1)$ and report $(B,t_1)$ yield, respectively,

\[
W_{t_1}^I(G,t_1) = (T - t_1 + p_0 \theta) \cdot \lim_{x \to +\infty} w(x),
\]

\[
W_{t_1}^I(B,t_1) = (T - t_1 + (1 - p_0) \theta) \cdot \lim_{x \to +\infty} w(x).
\]

We have assumed $p_0 \geq \frac{1}{2}$, and therefore $W_{t_1}^I(G,t_1) \geq W_{t_1}^I(B,t_1)$. Using [2], [8], and [9], one can calculate the flow payoff from staying silent until $t$, which equals $\left( \frac{b_0}{1 - b_0} (1 - F(t)) \right)$. Therefore, value from not making a report until the last point of $S$ evaluated at $t_1$ equals

\[
W_{t_1}^I(\emptyset) = \sum_{k=1}^{k=|S|} (t_{k+1} - t_k) \cdot w \left( \frac{b_0}{1 - b_0} (1 - F(t_k)) \right) + (T - \tilde{t} + \theta) \cdot w \left( \frac{b_0}{1 - b_0} (1 - F(\tilde{t})) \right).
\]
Therefore staying silent is optimal if and only if

\[
\lim_{x \to +\infty} w(x) \leq \sum_{k=1}^{k=|S|} \frac{t_{k+1} - t_k}{T - t_1 + p_0 \theta} \cdot w \left( \frac{b_0}{1 - b_0} \left(1 - F(t_k)\right) \right) + \frac{T - \bar{t} + \theta}{T - t_1 + p_0 \theta} \cdot w \left( \frac{b_0}{1 - b_0} \left(1 - F(\bar{t})\right) \right). \]

\[\square\]